J. Math. Anal. Appl. 421 (2015) 1519–1530

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Asymptotic behavior of global solutions of an anomalous diffusion system

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A R T I C L E I N F O

Article history:
Received 19 June 2014
Available online 8 August 2014
Submitted by J.J. Nieto

Keywords:
Parabolic system
Fractional Laplacian
Local and global existence
Asymptotic behavior

A B S T R A C T

This paper deals with an anomalous diffusion system which describes the spread of epidemics among a population. The analysis includes some results of the asymptotic behavior of global bounded solutions for this system with homogeneous Neumann boundary conditions.

1. Introduction

In this paper, we study the global existence and large time behavior of solutions to the system

$$\begin{cases} u_t = -a(-\Delta)^{\alpha} u - \lambda uv & \text{in } \mathbb{R}^+ \times \Omega \\ v_t = -b(-\Delta)^{\beta} v + \lambda uv - \mu v & \text{in } \mathbb{R}^+ \times \Omega \end{cases} \tag{1}$$

supplemented with the boundary and initial conditions

$$\begin{gathered} \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{in } \mathbb{R}^+ \times \partial\Omega, \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x) \quad \text{in } \Omega. \end{gathered} \tag{2}$$

Here Ω is a bounded regular domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ and u_0, v_0 are given nonnegative bounded functions, the constants a, b, λ and μ are positive.

This system describes the spread of epidemics within a confined population. The functions $u(x,t)$, $v(x,t)$ represent densities of susceptible and infected individuals. The positive constants λ and μ represent the

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http://dx.doi.org/10.1016/j.jmaa.2014.07.083

infection rate and the removal rate respectively (see [15]). The Neumann boundary conditions imply that there is no infection across the boundary. The presence of the nonlocal operator $(-\Delta)^\delta$ ($0 < \delta < 1$, $\delta = \alpha$ or β) which accounts for the anomalous diffusion [8,7] means that the sub-populations face some obstacles that slow their movement. In recent years, this phenomena of anomalous diffusion have been observed in many areas such that physics, finance and hydrology (see [12,13] and the references therein).

The system obtained from (1) by replacing the anomalous diffusion operator by the standard Laplacian operator $(-\Delta)$ was firstly studied in one-dimensional space by Webb [15]; he showed the existence of global bounded solutions and analyzed their behavior as t goes to $+\infty$.

Later on, a more general model was studied by Haraux and Kirane [3]. They took different diffusion coefficients for the two equations and general nonlinear terms. They proved the existence of global bounded solutions and investigated their asymptotic behavior. We can also mention the paper of Fitzgibbon and Morgan [2] who studied Webb's system in arbitrary bounded domain.

Recently, in [14], the authors developed a model of the spreading of an epidemic among three populations. The existence, uniform bounds and asymptotic behavior of solutions were investigated. An SIR epidemic system with equal diffusion coefficients was studied in [1]; local and global asymptotical stability was given by linearization and using a Lyapunov functional. These results were verified numerically. For other works on various types of the SIR epidemic model, the reader may be referred to [9,11] for example and the references therein.

Our model is a further extension as it contains super-diffusion terms.

The remainder of this paper is organized as follows. In Section 2, we present some definitions and preliminaries. In Section 3, we prove the existence of a local mild solution for the system (1)–(2). Finally, global existence and asymptotic behavior of solutions are studied in Section 4.

2. Preliminaries

Let us recall some facts that will be used in the sequel.

In an open bounded domain Ω, we denote by $(-\Delta_N)^\alpha$ the fractional power of the Laplacian in Ω with homogeneous Neumann boundary condition. Let λ_k ($k = 0, 1, \ldots, +\infty$) be the eigenvalues of the Laplacian operator in $L^2(\Omega)$ with homogeneous Neumann boundary condition and let φ_k be the corresponding eigenfunctions, *i.e.*

$$\begin{cases} (-\Delta_N)^\alpha \varphi_k = \lambda_k^\alpha \varphi_k & \text{in } \Omega, \\ \dfrac{\partial \varphi_k}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$D\big((-\Delta_N)^\alpha\big) = \left\{ u \in L^2(\Omega) \text{ such that } \frac{\partial u}{\partial \nu} = 0 \text{ and } \big\|(-\Delta_N)^\alpha u\big\|^2_{L^2(\Omega)} = \sum_{k=1}^{+\infty} \big|\lambda_k^\alpha \langle u, \varphi_k\rangle\big|^2 < +\infty \right\}.$$

So for $u \in D((-\Delta_N)^\alpha)$ we have

$$(-\Delta_N)^\alpha u = \sum_{k=1}^{+\infty} \lambda_k^\alpha \langle u, \varphi_k\rangle \varphi_k.$$

We have the following integration by parts formula

$$\int_\Omega u(x)(-\Delta_N)^\alpha v(x)\,dx = \int_\Omega v(x)(-\Delta_N)^\alpha u(x)\,dx, \quad \text{for } u, v \in D\big((-\Delta_N)^\alpha\big). \tag{3}$$