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Generalized Taylor series in hermitian Clifford analysis

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ABSTRACT

An orthogonal Appell basis of homogeneous polynomials is constructed for the Bergman space of square-integrable hermitian monogenic functions in the unit ball of $\mathbb{C}^n$, with values in a homogeneous subspace of spinor space. This construction then leads to a generalized Taylor series expansion for spherical monogenics.

1. Introduction

Let $\mathbb{B}_2$ be the open unit disc in the complex plane. The Bergman space

$$\mathcal{A}^2(\mathbb{B}_2) = \{ f(z) \in L^2(\mathbb{B}_2) : f \text{ is holomorphic in } \mathbb{B}_2 \}$$

is a separable Hilbert space for the traditional inner product

$$\langle f, g \rangle = \int_{\mathbb{B}_2} f(\zeta) \overline{g(\zeta)} \, dV(\zeta)$$

and possesses the reproducing kernel

$$K(z, \zeta) = \frac{1}{\pi} \frac{1}{(1 - z\overline{\zeta})^2}$$

i.e. for all $f \in \mathcal{A}^2(\mathbb{B}_2)$ it holds that

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$$f(z) = \int_{\mathbb{B}_2} K(z,\zeta) f(\zeta)\, dV(\zeta)$$

For this Bergman space $\mathcal{A}^2(\mathbb{B}_2)$ there exists a countable orthonormal basis $(\phi_j)_{j=0}^{\infty}$, and on each $E \times E$, E being a compact subset of $\mathbb{B}_2$, the series

$$\sum_{j=0}^{\infty} \phi_j(z)\overline{\phi_j(\zeta)}$$

uniformly converges to the reproducing kernel $K(z,\zeta)$. As is well-known, the natural powers $(z^j)_{j=0}^{\infty}$ form such a countable orthogonal basis for $\mathcal{A}^2(\mathbb{B}_2)$, and it may be directly verified that, indeed,

$$\sum_{j=0}^{\infty} \frac{j+1}{\pi} z^j \zeta^j = \frac{1}{\pi} \frac{1}{(1 - z\bar{\zeta})^2} = K(z,\zeta)$$

This orthogonal basis for $\mathcal{A}^2(\mathbb{B}_2)$ shows another important property: the derivative (with respect to the complex variable z) of a basis polynomial reproduces, up to a constant, another basis polynomial:

$$\left(z^j\right)' = j z^{j-1}$$

This property is known as the Appell property, see [1] for the original paper and [31,32,41] for the first contributions to systems of polynomials in a Clifford analysis setting. This Appell property is, quite naturally, important for numerical applications, since it facilitates formal and algorithmic manipulations of the basis polynomials without having to refer to their respective explicit forms.

In this paper we will construct an orthogonal Appell basis of homogeneous polynomials for a Bergman space of hermitian monogenic functions. Hermitian monogenic functions are one of the current research topics in Clifford analysis, which, in its most basic form, is a higher dimensional generalization of holomorphic function theory in the complex plane, and, at the same time, a refinement of harmonic analysis, see e.g. [9,34,27,36,35]. At the heart of Clifford analysis lies the notion of a monogenic function, i.e. a Clifford algebra valued null solution of the Dirac operator $\underline{\partial} = \sum_{\alpha=1}^{m} e_\alpha \partial_{X_\alpha}$, where $(e_1, \ldots, e_m)$ is an orthonormal basis of $\mathbb{R}^m$ underlying the construction of the real Clifford algebra $\mathbb{R}_{0,m}$. We refer to this setting as the Euclidean one, and to the Dirac operator as being rotation invariant, since the fundamental group action commuting with the Dirac operator $\underline{\partial}$ is the one of the special orthogonal group $\mathrm{SO}(m;\mathbb{R})$, which is doubly covered by the $\mathrm{Spin}(m)$ group of the Clifford algebra $\mathbb{R}_{0,m}$. In case the dimension m is even, say $m = 2n$, so-called hermitian Clifford analysis was recently introduced as a refinement of Euclidean Clifford analysis (see the books [44,25] and the series of papers [45,28,5,6,19,29,11]). The considered functions now take values in the complex Clifford algebra $\mathbb{C}_{2n}$ or in complex spinor space $\mathbb{S}_n$. Hermitian Clifford analysis is based on the introduction of an additional datum, a (pseudo) complex structure J, inducing an associated Dirac operator $\underline{\partial}_J$; it then focuses on the simultaneous null solutions of both operators $\underline{\partial}$ and $\underline{\partial}_J$, called hermitian monogenic functions. The fundamental group in this function theory, which still is in full development (see also [10,18,46,7,8,30,15]), is isomorphic with the unitary group $\mathrm{U}(n)$. It is worth mentioning that the traditional holomorphic functions of several complex variables are a special case of hermitian monogenic functions when the latter take their values in a specific homogeneous part of complex spinor space $\mathbb{S}_n$.

To meet the needs in numerics, recently much effort has been put into constructing orthogonal bases for spaces of homogeneous monogenic polynomials, called (solid) spherical monogenics, both in the framework of Euclidean and of hermitian Clifford analysis. The basis polynomials arising in the Taylor series expansion of (standard) monogenic functions, sometimes called Fueter polynomials, are not suitable for that purpose since they are not orthogonal with respect to the Fischer inner product, which is the natural inner product on Clifford algebra valued polynomials. Explicit constructions of orthogonal polynomial bases in the Euclidean