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Existence of periodic solutions of a particular type of super-quadratic Hamiltonian systems ☆

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ABSTRACT

This paper considers the Hamiltonian systems with new super-quadratic conditions covering the case $\hat{H}(t,z) = (|p|^{1+\frac{\mu}{\nu}} + |q|^{1+\frac{\nu}{\mu}})(\ln(1+|z|^{\hat{\xi}}))^{\hat{\eta}}$, where $\hat{\xi}, \hat{\eta}, \mu, \nu > 1$ with $\frac{1}{\mu} + \frac{1}{\nu} < 1$ and $\frac{\mu}{2\nu} < \frac{\nu}{\mu} < \frac{\mu}{\nu} < 1 + \frac{\nu}{\mu}$. Using the mini-max principle, we obtain the existence of the periodic solutions.

1. Introduction

Let $z = (p, q)$, $p, q \in \mathbf{R}^n$ and $H \in C^1(\mathbf{R} \times \mathbf{R}^{2n}, \mathbf{R})$, then we consider the Hamiltonian system

$$\begin{cases} \dot{p} = -H'_q(t,z), \\ \dot{q} = H'_p(t,z), \end{cases} \tag{1.1}$$

which also can be written as $\dot{z} = JH'_z(t,z)$, where $H'_z = \frac{\partial H}{\partial z} = (H'_p, H'_q) = (\frac{\partial H}{\partial p}, \frac{\partial H}{\partial q})$ and $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ with I_n being the $n \times n$ identity matrix.

In the pioneer work [5], the author established the existence of periodic solutions of the autonomous Hamiltonian systems under a classical super-quadratic condition, that is,
(S_1) there exist constants $\hat{\theta} \in (0, \frac{1}{2})$ and $R > 0$ such that

$$\hat{\theta} H'_z(t,z) \cdot z \geq H(t,z) > 0, \quad (t,z) \in \mathbf{R} \times \mathbf{R}^{2n} \text{ with } |z| \geq R.$$

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After paper [5], much work has been done in this field. Paper [1] develops a mini-max principle which improves the verification of the existence of periodic solutions in paper [5]. Paper [2] considers periodic and subharmonic solutions of the super-quadratic non-autonomous Hamiltonian systems. Paper [3] considers the existence of periodic solutions of a certain kind of the Hamiltonian systems with the super-quadratic condition (S_2) generalizing (S_1), that is,
(S_2) there exist constants c_1, c_2, $R > 0$ and $\hat{\beta} > 1$ such that

$$H_z'(t, z) \cdot z - 2H(t, z) \geq c_1 |z|^{\hat{\beta}}, \quad (t, z) \in \mathbf{R} \times \mathbf{R}^{2n} \text{ with } |z| \geq R,$$
$$\frac{H(t, z)}{|z|^2} \to +\infty \quad \text{as } |z| \to +\infty \text{ uniformly in } t,$$

where "$\cdot$" denotes the inner product in $\mathbf{R}^{2n}$ and "$|\cdot|$" denotes the induced norm.

Paper [4] considers the existence of periodic solutions of the Hamiltonian systems with the super-quadratic condition (S_3) generalizing (S_1), that is,
(S_3) there exist constants $\hat{\sigma}, \hat{\tau} > 1$ with $\frac{1}{\hat{\sigma}} + \frac{1}{\hat{\tau}} < 1$ and $R > 0$ such that

$$\frac{1}{\hat{\sigma}} H_p'(t, z) \cdot p + \frac{1}{\hat{\tau}} H_q'(t, z) \cdot q \geq H(t, z) > 0, \quad (t, z) \in \mathbf{R} \times \mathbf{R}^{2n} \text{ with } |z| \geq R.$$

Paper [4] also shows that (S_3) implies that there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$H(t, z) \geq c_1 \left(|p|^{\hat{\sigma}} + |q|^{\hat{\tau}} \right) - c_2, \quad (t, z) \in \mathbf{R} \times \mathbf{R}^{2n}. \tag{1.2}$$

References [7] and [6] are good readings for the beginners to acquire the knowledge in this field.

Our aim is using the mini-max method to find out a nontrivial periodic solution of the system (1.1) with a new super-quadratic condition generalizing (S_2) and (S_3). Claim 1.7 implies that there exist particular cases that satisfy the super-quadratic condition in this paper, but dissatisfy (S_2) and (S_3).

Our results are listed as follows.

Theorem 1.1. *The system (1.1) possesses a nontrivial T-periodic solution, if H satisfies*

(H1) $H \in C^1(\mathbf{R} \times \mathbf{R}^{2n}, \mathbf{R})$ *is nonnegative and T-periodic with respect to t,*
(H2) *there exist constants $c_1, c_2 > 0$ and β, μ, $\nu > 1$ with $\frac{1}{\mu} + \frac{1}{\nu} < 1$ such that*

$$\frac{1}{\mu} H_p'(t, z) \cdot p + \frac{1}{\nu} H_q'(t, z) \cdot q - \left(\frac{1}{\mu} + \frac{1}{\nu} \right) H(t, z) \geq c_1 |z|^\beta - c_2, \quad (t, z) \in \mathbf{R} \times \mathbf{R}^{2n},$$

(H3) *there exist constants $\sigma, \tau > 0$ and $\lambda \in (\max\{\frac{\sigma}{\tau}, \frac{\tau}{\sigma}\}, 1 + \frac{\beta - 1}{\beta})$ such that*

$$\left| H_z'(t, z) \right| \leq c_2 \left(|z|^\lambda + 1 \right), \quad (t, z) \in \mathbf{R} \times \mathbf{R}^{2n},$$

where c_2 and β are as above,
(H4) $\frac{H(t,z)}{|p|^{1+\frac{\sigma}{\tau}} + |q|^{1+\frac{\tau}{\sigma}}} \to 0$ *as $|z| \to 0$ uniformly in t, where σ and τ are as above,*
(H5) $\frac{H(t,z)}{|p|^{1+\frac{\sigma}{\tau}} + |q|^{1+\frac{\tau}{\sigma}}} \to +\infty$ *as $|z| \to +\infty$ uniformly in t, where σ and τ are as above.*

Note that (H2) and (H5) are the super-quadratic conditions that generalize (S_2) and (S_3) (see Claim 1.2). And there exists an example showing that it satisfies (H1)–(H5), but dissatisfies (S_2) and (S_3) (see Claim 1.7). So Theorem 1.1 generalizes Theorem 1.1 in paper [3].