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Numerical approximations for highly oscillatory Bessel transforms and applications ☆

Ruyun Chen

College of Science, Guangdong Ocean University, Zhanjiang, Guangdong 524088, China

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ABSTRACT

This paper presents an efficient numerical method for approximating highly oscillatory Bessel transforms. Based on analytic continuation, we transform the integrals into the problems of integrating the forms on $[0, +\infty)$ with the integrand that does not oscillate and decays exponentially fast, which can be efficiently computed by using Gauss–Laguerre quadrature rule. We then derive the error of the method depending on the frequency and the node number. Moreover, we apply the scheme for studying the approximations of the solutions of two kinds of highly oscillatory integral equations. Preliminary numerical results show the efficiency and accuracy of numerical approximations.

1. Introduction

The problem of evaluating the highly oscillatory Bessel transforms

$$I[f] = \int_0^a f(x) J_v(\omega x) dx, \quad \omega \gg 1 \tag{1.1}$$

occurs in many areas of science and technology, for example, in astronomy, optics, electro-magnetics, seismology image processing and applied mathematics (see [3,4,10,19,31,32]). Here, $f(x)$ is a smooth real-valued function, v is an arbitrary nonnegative integer and $J_v(\omega x)$ is the Bessel function of the first kind and of order v. In most of the cases, the integrals cannot be done analytically and one has to rely on numerical methods. However, for large ω, the integrands become highly oscillatory and present serious difficulties in obtaining numerical convergence of the integrations. For example, classical quadrature rules of the Newton–Cotes type or Gaussian type are based on polynomial interpolation. It is well known that polynomials are

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E-mail address: csuchenruyun@yahoo.com.cn.

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not suited for the approximation of oscillatory functions, and the integration error of these quadrature rules increases rapidly with increasing ω.

The theoretical and numerical aspects of the integrals have received considerable attention in the last few years and many efficient methods have been developed. Modified Clenshaw–Curtis methods [29] are applicable to the integral $\int_0^1 f(x)J_v(\omega x)dx$ for non-negative order v, which is presented by replacing the smooth factor of the integrand by a truncated Chebyshev series approximation. Levin-type methods (see [22, 23,28,38,41]) are developed for evaluating $\int_a^b f(x)J_v(\omega x)dx$ under the assumption that the weight functions satisfies certain differential conditions for $0 \notin [a,b]$. Based on Lagrange's identity

$$z\mathcal{L}S - S\mathcal{M}z = \big(\mathcal{Z}\big(S(\omega,x), z(x)\big)\big)'_x \tag{1.2}$$

by choosing the differential operator $\mathcal{L}$ to satisfy $\mathcal{L}S = 0$ for the weight function $S(\omega,x)$, where $\mathcal{M}$ is the adjoint operator of $\mathcal{L}$, generalized quadrature rules are presented under the condition that $0 \notin [a,b]$. This method is efficient for more general irregular oscillatory weight functions $S(\omega,x)$. But a disadvantage is that the approach only works for $0 \notin [a,b]$ [9,13,40]. Based on He [16,17] and Watson's work [35,36], Chen and Liang proposed a homotopy perturbation method to consider the asymptotic expression of the Bessel, Anger and Weber transformations [7,8]. All these methods share an advantageous property that the accuracy greatly improves when ω increases.

Complex integration method [27,37] and steepest descent method [18] for the form

$$\int_{-1}^{1} f(x)e^{i\omega x}dx \tag{1.3}$$

have been developed. If f is analytic in a sufficiently large complex region G that contains $[-1,1]$, the integrals can be transformed into the problems of integrating two integrals on $[0,+\infty)$, where the integrands are not oscillatory and decay exponentially fast [18,21,27].

In the present work, an entirely different method is presented to compute the integrals (1.1). We consider the numerical computation of the integral following Wong [37], Milovanović [27] and Huybrechs and Vandewalle's work [18].

The numerical solution of the integral equation of the first kind with a highly oscillatory Bessel kernel

$$\int_0^x J_0\big(\omega(x-t)\big)y(t)dt = f(x), \quad x \in [0,T], \tag{1.4}$$

has attracted much attention during the last few years [6,10,30,33], where $y(x)$ is the unknown function whose value is to be determined in the interval $[0,T]$, $f(x)$ is a given sufficiently smooth function and ω is a large parameter.

One feature of the Volterra integral equation (1.4) is of particular note: when $\omega \gg 1$, the kernel function $J_0(\omega(x-t))$ would become highly oscillatory. This means that some general numerical methods may not be immediately applicable to this equation. When $f(x) \in C^1[0,T]$, the unique solution of Eq. (1.4) can be given by [30, Eq. 1.8.1] as

$$y(x) = \frac{1}{\omega}\left(\frac{d^2}{dx^2} + \omega^2\right)^2 \int_0^x (x-t)J_1\big(\omega(x-t)\big)f(t)dt, \quad x \in [0,T]. \tag{1.5}$$

At the same time, we will consider the numerical solution of the Fredholm integral equation with a highly oscillatory Bessel kernel