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Logarithmic moving averages

N.H. Bingham [a], Bujar Gashi [b,*]

[a] *Department of Mathematics, Imperial College London, 180 Queens Gate, London, SW7 2BZ, UK*
[b] *Department of Mathematical Sciences, The University of Liverpool, Liverpool, L69 7ZL, UK*

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We introduce a moving average summability method, which is proved to be equivalent to the logarithmic ℓ-method. Several equivalence and Tauberian theorems are given. A strong law of large numbers is also proved.

1. Introduction

The logarithmic methods of summation ℓ and L are classical (see, for example, Hardy [35], Ishiguro [37–40], Kwee [46–48]). Let $\{s_n\}_{n=0}^{\infty}$ be a sequence of real numbers. The sequence is summable to s by the logarithmic ℓ-method, written $s_n \to s$ (ℓ), if

$$t_n := \frac{1}{\log n} \sum_{i=0}^{n} \frac{s_i}{i+1} \to s \quad (n \to \infty) \tag{1.1}$$

(we write ℓ_x when the limit is taken through a continuous variable).

The sequence is summable to s by the logarithmic L-method, written $s_n \to s$ (L), if

$$\frac{1}{-\log(1-x)} \sum_{i=0}^{\infty} \frac{s_i}{i+1} x^{i+1} \to s \quad (x \uparrow 1). \tag{1.2}$$

* Corresponding author.
E-mail addresses: N.Bingham@imperial.ac.uk (N.H. Bingham), Bujar.Gashi@liverpool.ac.uk (B. Gashi).

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Here we introduce a certain delayed (deferred) summability method. For $\lambda > 1$, the sequence $\{s_n\}_{n=0}^{\infty}$ is summable by the *logarithmic moving average*, $s_n \to s$ $(\mathcal{L}(\lambda))$, if

$$\frac{1}{\log n} \sum_{n^{1/\lambda} < i \leq n} \frac{s_i}{i+1} \to \left(1 - \lambda^{-1}\right) s \quad (n \to \infty) \tag{1.3}$$

(we write $\mathcal{L}_x(\lambda)$ if the limit is taken through a continuous variable x).

2. Equivalence and Tauberian theorems

2.1. Results

Theorem 1. *$\ell \Leftrightarrow \mathcal{L}(\lambda)$ for some (all) $\lambda > 1$.*

With $\{a_n\}_{n=0}^{\infty}$ defined by $s_n = \sum_{k=0}^{n} a_n$, the Riesz (typical) mean $R(\log n)$ (of order 1) is defined as

$$\frac{1}{x} \int_0^x \left\{ \sum_{k:\ \log(k+1) < y} a_k \right\} dy \quad (x \to \infty).$$

In view of $R(\log n) \Leftrightarrow \ell$ (Hardy [35, Th. 37]; see also §5.16), this gives

Corollary 1. *$R(\log n) \Leftrightarrow \mathcal{L}(\lambda)$ for some (all) $\lambda > 1$.*

Note that $R(\log n)$ involves a *continuous* limit, but $\mathcal{L}(\lambda)$ a *discrete* one. This equivalence between discrete and continuous limits is a consequence of *uniformity*, as in Theorem 2 below.

Theorem 2. *If (1.3) holds for all $\lambda > 1$, then it holds uniformly on compact λ-sets in $(1, \infty)$.*

Corollary 2. *$s_n \to s$ (ℓ_x) if and only if $s_n \to s$ $(\mathcal{L}_x(\lambda))$ for all $\lambda > 1$.*

Theorem 3. *Let $U(x) := \sum_{0 \leq i \leq x} s_i (i+1)^{-1}$. The following statements are equivalent:*

(i) *$U(x) = U_1(x) - U_2(x)$, with $U_2(x)$ non-decreasing and $U_1(x)$ satisfying*

$$\lim_{x \to \infty} \left[U_1(x) - U_1\left(x^{1/\lambda}\right)\right] (\log x)^{-1} = s\left(1 - \lambda^{-1}\right), \quad \forall \lambda > 1,$$

(ii) $\liminf_{\alpha \downarrow 1} \limsup_{x \to \infty} \sup_{\theta \in [1, \alpha]} [U(x) - U(x^{1/\theta})](\log x)^{-1} < \infty$.

Corollary 3. *If $s_n \to s$ (ℓ), then statement (ii) of Theorem 3 holds.*

The Abelian result that $\ell \Rightarrow L$ is proved in [38]. The simplest Tauberian condition for $L \Rightarrow \ell$, and thus $\ell \Leftrightarrow L$, is $s_n = O_L(1)$ as proved in [39]. We next give a Tauberian theorem establishing the equivalence between ℓ and L methods under a one-sided Tauberian condition of best possible character (cf. [16]).

Theorem 4. *We have $\ell \Leftrightarrow L$ if and only if*

$$\lim_{\lambda \downarrow 1} \liminf_{n \to \infty} \min_{n \leq m \leq \lambda n} \frac{1}{\log n} \sum_{n \leq i \leq m} \frac{s_i}{i+1} \geq 0. \tag{2.1}$$