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Rectifying curves in the three-dimensional sphere

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Rectifying curves in $\mathbb{R}^3$ were introduced by B.Y. Chen in 2003 as space curves whose position vector always lies in its rectifying plane. In this paper, we extend this definition, as well as several results about rectifying curves obtained by Chen and Dillen (2005) and Izumiya and Takeuchi (2004), to curves in the three-dimensional sphere. Our results give nice and relevant differences between spherical and Euclidean geometries.

1. Introduction

Let $x : I \to \mathbb{R}^3$ be a unit speed space curve, where I is an open interval, such that $x''(s)$ is never zero. Let $\{T = x', N, B = T \times N\}$ be the Frenet frame, whose derivatives are given by the well known Frenet–Serret equations:

$$T' = \kappa N, \qquad N' = -\kappa T + \tau B, \qquad B' = -\tau N, \tag{1}$$

where $\kappa > 0$ and τ are called the curvature and torsion of the curve, respectively. At each point of the curve, the plane spanned by $\{T, B\}$ is called the rectifying plane. Chen defined in [3] (see also [4]) rectifying curves in $\mathbb{R}^3$ as curves whose position vector always lies in its rectifying plane. More precisely, a curve x is said to be a rectifying curve if there exists a point p in $\mathbb{R}^3$ such that

$$x(s) - p = \lambda(s)T(s) + \mu(s)B(s), \tag{2}$$

for some differentiable functions λ and μ. The fixed point p is precisely the point that belongs to all rectifying planes. In the previous equation, we simultaneously use the double nature of the Euclidean space $\mathbb{R}^3$: on the one hand, as the differentiable manifold where the curve lies, and, on the other hand, as the tangent vector

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space at any point of the curve. Since Chen's paper, many authors have extended the notion of rectifying curve to other ambient spaces (of dimension $n \geq 3$), endowed with a Riemannian or pseudo-Riemannian metric (see, e.g., [1,6–12,16]). In all cases, the manifold and the tangent vector space can be identified from each other, and this identification allows one to perform an analogous study to that made by Chen.

However, if we want to extend this concept to other ambient spaces, it is necessary to distinguish between the manifolds and their tangent vector spaces. In this regard, the key of Eq. (2) is that the straight line that connects $x(s)$ with the point p is orthogonal to the principal normal line (i.e., the line starting at $x(s)$ in the direction of $N(s)$). This idea will be used to define rectifying curves in the three-dimensional sphere $\mathbb{S}^3(r)$. Naturally, a similar approach can be done to study rectifying curves in the three-dimensional hyperbolic space $\mathbb{H}^3(-r)$ or even in ambient spaces endowed with an indefinite metric (in particular, three-dimensional Lorentzian spaces).

This paper is organized as follows. After a section devoted to some basic preliminaries, we introduce the concept of rectifying curve in the three-dimensional sphere (Definition 1). In Section 3, we prove that a curve γ in $\mathbb{S}^3(r)$ is a rectifying curve if and only if γ is a geodesic of a conical surface (Theorem 1). In Section 4, we present a nice characterization for rectifying curves. In fact, we prove that a unit speed curve $\gamma = \gamma(s)$ is a rectifying curve if and only if the ratio of torsion and curvature is given by $\tau/\kappa = c_1 \sin((s + s_0)/r) + c_2 \cos((s + s_0)/r)$, for some constants c_1, c_2 and s_0 (Theorem 2). In Section 5, we provide some simple characterizations for twisted rectifying curves (Theorem 4). In Section 6, we give a classification result for rectifying curves in the sphere: a twisted curve $\gamma = \gamma(t)$ is a rectifying curve if and only if $\gamma(t) = \exp_p(\rho(t)V(t))$, where V is a unit speed curve in $\mathbb{S}^2(1) \subset T_p\mathbb{S}^3(r)$ and $\rho(t) = r \arctan(a \sec(t + t_0))$, for some constants $a \neq 0$ and t_0 (Theorem 5). In the last section, we study the rectifying developable of a curve and show that the rectifying developable of a unit speed curve γ is a conical surface if and only if γ is a rectifying curve (Theorem 7).

2. Preliminaries

Let $\mathbb{S}^3(r)$ denote the three-dimensional sphere in $\mathbb{R}^4$ of radius $r > 0$ and centered at the origin, defined by

$$\mathbb{S}^3(r) = \left\{ x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \;\middle|\; \langle x, x\rangle = \sum_{i=1}^{4} x_i^2 = r^2 \right\}.$$

Let $\overline{\nabla}$ and ∇^0 denote the Levi-Civita connections on $\mathbb{S}^3(r)$ and $\mathbb{R}^4$, respectively. If X and Y are vector fields tangent to $\mathbb{S}^3(r)$, then $\overline{\nabla}$ and ∇^0 are related by the Gauss formula as follows

$$\nabla^0_X Y = \overline{\nabla}_X Y - \frac{1}{r^2}\langle X, Y\rangle y,$$

where $y : \mathbb{S}^3(r) \to \mathbb{R}^4$ denotes the position vector.

Consider a unit speed curve $\gamma : I \to \mathbb{S}^3(r)$, where I is a real open interval or $\mathbb{S}^1$, and assume that γ is not a geodesic curve. Let $T_\gamma(s)$ denote $\gamma'(s)$. Then there exists a unique vector field $N_\gamma(s)$ and a positive function $\kappa_\gamma(s)$ so that $\overline{\nabla}_{T_\gamma} T_\gamma = \kappa_\gamma N_\gamma$; here, $\overline{\nabla}_{T_\gamma} T_\gamma$ denotes the covariant derivative of T_γ along γ in $\mathbb{S}^3(r)$. N_γ is called the principal normal vector field and κ_γ the curvature of the given curve.

In the sphere $\mathbb{S}^3(r)$ we can define a cross product as follows. Consider a point $q \in \mathbb{S}^3(r)$ and take two tangent vectors $v_1, v_2 \in T_q\mathbb{S}^3(r)$. The cross product of v_1 and v_2 is the unique tangent vector $v_1 \times v_2$ in $T_q\mathbb{S}^3(r)$ such that

$$\langle v_1 \times v_2, w\rangle = \det(v_1, v_2, w, q), \quad \forall w \in T_q\mathbb{S}^3(r),$$