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Positive solutions for the Hardy–Sobolev–Maz'ya equation with Neumann boundary condition

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A R T I C L E I N F O

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A B S T R A C T

Let Ω be a bounded domain with a smooth C^2 boundary in $\mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^{N-k}$ $(N \geq 3)$, $0 \in \partial\Omega$, and ν denotes the unit outward normal vector to boundary $\partial\Omega$. We are concerned with the Neumann boundary problem: $-\Delta u - \mu\frac{u}{|y|^2} = \frac{|u|^{p_t-1}u}{|y|^t} + f(x,u)$, $u > 0$, $x \in \Omega$, $\frac{\partial u}{\partial\nu} + \alpha(x)u = 0$, $x \in \partial\Omega \setminus \{0\}$. Using the Mountain Pass Lemma without (PS) condition and the strong maximum principle, we establish certain existence result of the positive solutions.

1. Introduction and main results

In this paper, we consider the following Hardy–Sobolev–Maz'ya problem with Neumann boundary

$$\begin{cases} -\Delta u - \mu\dfrac{u}{|y|^2} = \dfrac{|u|^{p_t-1}u}{|y|^t} + f(x,u), & u > 0,\ x \in \Omega, \\ \dfrac{\partial u}{\partial\nu} + \alpha(x)u = 0, & x \in \partial\Omega \setminus \{0\}, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary, $0 \in \partial\Omega$, $\mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^{N-k}$, $2 \leq k < N$, $0 \leq \mu < \bar{\mu} =: \frac{(k-2)^2}{4}$ when $k > 2$, $\mu = 0$ when $k = 2$, $0 \leq t < 2$ and $p_t = \frac{N+2-2t}{N-2}$, moreover, $0 \leq \alpha(x) \in L^\infty(\partial\Omega)$, $\alpha(x) \not\equiv 0$, and ν denotes the unit outward normal vector to boundary $\partial\Omega$. The main interest of this kind of problem, in addition to the presence of the singular potential $1/|y|^2$ related to Hardy's inequality, is the following well-known Hardy–Sobolev–Maz'ya inequality [8]

$$\left(\int\limits_{\mathbb{R}^N} \frac{|u|^{p_t+1}}{|y|^t}dx\right)^{\frac{2}{p_t+1}} \leq \left(S_t^\mu\right)^{-1} \int\limits_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu\frac{u^2}{|y|^2}\right)dx, \quad \forall u \in C_0^\infty\left(\mathbb{R}^N\right), \tag{1.2}$$

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where $\mu < \bar{\mu}$. Especially, when $t = 2$, one has that if $u \in H_0^1(\Omega)$, then $u/|y| \in L^2(\Omega)$ and

$$\int\limits_\Omega \frac{u^2}{|y|^2} dx \leq \frac{1}{\bar{\mu}} \int\limits_\Omega |\nabla u|^2 dx.$$

But, the constant $\bar{\mu}$ is optimal and is not achieved, due to the non-compactness in the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega, |y|^{-2}dx)$, even locally, in any neighborhood of zero.

From the mathematical point of view, the Hardy–Sobolev–Maz'ya terms $\frac{u}{|y|^2}$ and $\frac{|u|^{p_t-1}u}{|y|^t}$ are critical: indeed they have the same homogeneity as the Laplacian and do not belong to the Kato's class, hence they can cause the non-compactness of the corresponding functional. Various equations similar to (1.1) have been proposed to model several phenomena of interest in astrophysics. We recall here the Eddingtons and Matukumas equations, which have attracted much interest in recent years (see [4,22–24]). In [7] various astrophysical models are introduced and discussed, including some generalizations of the Matukumas equation.

Consider the functional associated to (1.1)

$$I(u) = \frac{1}{2} \int\limits_\Omega \left(|\nabla u|^2 - \lambda \frac{u^2}{|y|^2} \right) dx - \frac{1}{p_t + 1} \int\limits_\Omega \frac{(u^+)^{p_t+1}}{|y|^t} dx - \int\limits_\Omega F(x, u^+) dx + \frac{1}{2} \int\limits_{\partial\Omega} \alpha(x) u^2 d\sigma, \tag{1.3}$$

where $u^+ = \max\{u, 0\}$, $F(x, s) = \int_0^s f(x, t)dt$. Obviously, the functional I is well defined and is C^1 smooth in $H^1(\Omega)$, and hence by the strong maximum principle, its critical point is a weak solution of (1.1). In what follows, we will employ the minimax methods to find the nontrivial critical points of I. The main difficulty is that, owing to the non-compactness in the embedding $H^1(\Omega) \hookrightarrow L^{p_t+1}(\Omega, |y|^{-t}dx)$, I does not satisfy the Palais–Smale compactness condition for large level, which makes the study of (1.1) very tough.

After the pioneer work of Brezis and Nirenberg [10], a large number of papers appeared concerning the Dirichlet problem with critical exponents when $k = N$ (see [5,6,9,11–14,17,19,20]). The corresponding problem with Neumann boundary condition have also been extensively studied in recent years (see for example, [1–3,16,21,25,29,30]). As for the Sobolev–Hardy problem, since the singularity at 0, this kind of problem attracts much attention. However, there are few interesting results for this kind of problem with $k \neq N$. The main purpose of this paper is to establish new existence phenomena for problem (1.1) under the case $t = 2 - \frac{N-2}{N-k+\sqrt{(k-2)^2-4\mu}}$.

For $k = N$, the existence of the least energy solutions of (1.1) has been proved in [15] if $\mu < (\frac{N-2}{2})^2$ and $0 < t < 2$ for $N \geq 3$ with Dirichlet boundary condition and $f(x, u) = 0$; Han and Liu in [18] showed the existence of positive solutions of (1.1) when $t = 0$ for the Neumann boundary condition with $0 \in \partial\Omega$; Shang and Tang in [27] proved that (1.1) has a positive solution if $0 < \mu < (\frac{N-2}{2})^2$ and $0 \leq t < 2$ for $N \geq 4$. If $2 < k < N$, Bhakta and Sandeep in [8] investigated the existence (non-existence) of the solutions to (1.1) with Dirichlet boundary condition; replacing $f(x, u)$ by λu, the authors in [28] obtained infinitely many solutions for (1.1) when $N > 6 + t$. In this sequel, using the variational method and taking some idea from [18,27], we prove that (1.1) possesses a positive solution for a large class of $f(x, s)$.

In order to state our main result, we give the conditions imposed on $f \in C(\bar{\Omega} \times \mathbb{R}^+, \mathbb{R})$: there exists a function $a(x) \in L^\infty(\Omega)$, $a(x) \leq 0$ such that

(f_1) $\lim_{s\to 0^+} \frac{f(x,s)}{s} = a(x)$, uniformly for $x \in \bar{\Omega}$;
(f_2) $\lim_{s\to +\infty} \frac{f(x,s)}{s^{p_t}} = 0$, uniformly for $x \in \bar{\Omega}$.