J. Math. Anal. Appl. 418 (2014) 569–594

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A phase decomposition approach and the Riemann problem for a model of two-phase flows

Mai Duc Thanh

Department of Mathematics, International University, Quarter 6, Linh Trung Ward, Thu Duc District, Ho Chi Minh City, Viet Nam

ARTICLE INFO

Article history:
Received 24 January 2013
Available online 13 April 2014
Submitted by H. Liu

Keywords:
Two-phase flow
Nonconservative
Phase decomposition
Generalized Rankine–Hugoniot relations
Shock wave
Riemann problem

ABSTRACT

We present a phase decomposition approach to deal with the generalized Rankine–Hugoniot relations and then the Riemann problem for a model of two-phase flows. By investigating separately the jump relations for equations in conservative form in the solid phase, we show that the volume fractions can change only across contact discontinuities. Then, we prove that the generalized Rankine–Hugoniot relations are reduced to the usual form. It turns out that shock waves and rarefaction waves remain on one phase only, and the contact waves serve as a bridge between the two phases. By decomposing Riemann solutions into each phase, we show that Riemann solutions can be constructed for large initial data. Furthermore, the Riemann problem admits a unique solution for an appropriate choice of initial data.

1. Introduction

We consider in this paper the Riemann problem for the following model of two-phase flows, see [6],

$$
\begin{aligned}
&\partial_t(\alpha_g\rho_g) + \partial_x(\alpha_g\rho_g u_g) = 0,\\
&\partial_t(\alpha_g\rho_g u_g) + \partial_x\big(\alpha_g\big(\rho_g u_g^2 + p_g\big)\big) = p_g\partial_x\alpha_g,\\
&\partial_t(\alpha_s\rho_s) + \partial_x(\alpha_s\rho_s u_s) = 0,\\
&\partial_t(\alpha_s\rho_s u_s) + \partial_x\big(\alpha_s\big(\rho_s u_s^2 + p_s\big)\big) = -p_g\partial_x\alpha_g,\\
&\partial_t\rho_s + \partial_x(\rho_s u_s) = 0, \quad x\in\mathbb{R}, t>0.
\end{aligned}
\tag{1.1}
$$

The system (1.1) is obtained from the full model of two-phase flows, see [4,6], namely the gas phase and the solid phase, by assuming that the flow is isentropic in both phases. The first and the third equation of (1.1) describe the conservation of mass in each phase; the second and the fourth equation of (1.1) describe

E-mail address: mdthanh@hcmiu.edu.vn.

http://dx.doi.org/10.1016/j.jmaa.2014.04.012
0022-247X/

the balance of momentum in each phase; the last equation of (1.1) is the so-called compaction dynamics equation. Throughout, we use the subscripts g and s to indicate the quantities in the gas phase and in the solid phase, respectively. The notations $\alpha_k, \rho_k, u_k, p_k, k = g, s$, respectively, stand for the volume fraction, density, velocity, and pressure in the k-phase, $k = g, s$. The volume fractions satisfy

$$\alpha_s + \alpha_g = 1. \tag{1.2}$$

The system (1.1) has the form of nonconservative systems of balance laws

$$U_t + A(U)U_x = 0, \tag{1.3}$$

for $U = (\rho_g, u_g, \rho_s, u_s, \alpha_g)$, and $A(U)$ is given at the beginning of Section 2. Weak solutions of such a system can be understood in the sense of *nonconservative products* – a concept introduced by Dal Maso, LeFloch and Murat [7]. In Section 2 we will brief this concept. Nonconservative systems can be used to model multi-phase flows. Multi-phase flow models have attracted attention of many scientists not only for the study theoretical problems such as existence, uniqueness, stability, and constructions of solutions, but also numerical approximations of the solutions. In the case of two-phase flow models, there may be two classes: the class of one-fluid models of two-phase flows and the class of two-fluid models of two-phase flows. The situation is similar to multi-phase flow models. Both classes are of nonconservative form, but there is a major difference between the two. That is, one-pressure models of two-phase flows are in general not hyperbolic (see [12]), but two-pressure models of two-phase flows such as (1.1) are hyperbolic and strictly hyperbolic except on a finite number of hyper-surfaces of the phase domain. Moreover, the characteristic fields of two-phase flow models such as (1.1) have explicit forms. This raises the hope for the study of the two-pressure models of two-phase flows: the theory of shock waves for hyperbolic systems of conservation laws may be developed to study these hyperbolic models. In the numerical approximations, numerical schemes employing an explicit form of characteristic fields, such as Roe-type schemes, can be implemented. Therefore, the model (1.1), having applications in the modeling of the deflagration-to-detonation transition (DDT) in granular explosives, is worth to study.

In this paper, we will present a method to construct solutions of the Riemann problem for the two-phase flow model (1.1) using a phase decomposition approach. Observe that the construction of solutions of the Riemann problem for hyperbolic systems when the solution vector has dimension larger than three is in general very complicated. The solution vector of (1.1) has dimension five, so how to work through? It is very interesting that in the system (1.1), four characteristic fields depend only on one phase, either gas or solid. This allows the waves associated with these characteristic fields to change only in one phase and remain constants in the other. This motivates us to propose a phase decomposition approach to build Riemann solutions of (1.1), where waves associated with the 5th characteristic field will then be used as a "bridge" to connect between the two phases. Precisely, we first investigate the impact of the jump conditions of the conservation equations in the solid phase. This leads us to a crucial conclusion that across a discontinuity, either the volume fractions remain constant, or the discontinuity is a contact wave. The first case yields the usual form of the jump relations for shock waves. In the later case we can also point out that the jump relations have the canonical form, by using a regularization method. Then, by a decomposition technique, we can separate the Riemann solutions in each phase, where the two phases are now constraint to each other via the solid contact. So, one can see that the solid contact in this method serves as a "bridge" to connect between the group of solid components and the group of gas components of the Riemann solutions. As usual, Riemann solutions are made from fundamental waves: shock waves, contact discontinuities, and rarefaction waves. Shock waves and contact discontinuities are weak solutions in the sense of nonconservative products and are of the usual form