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# Finite time blow-up and global solutions for fourth order damped wave equations

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ABSTRACT

This work is devoted to a class of fourth order wave equations with linear damping term and superlinear source term. After showing the uniqueness and existence of local solutions to the equations, we give necessary and sufficient conditions for global existence and finite time blow-up of these solutions. Moreover, the potential well depth is estimated.



## 1. Introduction

The report [1] about the Tacoma Narrows Bridge collapse [25,26] considers *. . . the crucial event in the collapse to be the sudden change from a vertical to a torsional mode of oscillation*, see also [25, p. 63]. Hence, if one models a suspension bridge by a beam, there is no way to highlight the torsional oscillations. The nonlinear behavior of suspension bridges, which is by now well established, see [2,8,13,23], also plays a crucial role in causing oscillations. Therefore, a reliable model for suspension bridges should be nonlinear and it should have enough degrees of freedom to display torsional oscillations. In this respect, Lazer and McKenna [14, Problem 11] suggested to study the following equation

$$\Delta^2 u + c^2 \Delta u + h(u) = 0, \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $h(u) \approx [u+1]^+ - 1$ with $u^+ = \max\{u, 0\}$. Subsequently, equations "like" (1.1), namely,

$$\Delta^2 u + c^2 \Delta u = b\big[(u+1)^+ - 1\big], \quad \text{in } \Sigma \subset \mathbb{R}^n, \tag{1.2}$$

under the Navier boundary condition have been considered in several papers. For the case $c^2 < \lambda_1$ ($(\lambda_k)_{k\geq 1}$, the sequence of eigenvalues of $-\Delta$ in $H_0^1(\Sigma)$), we refer to [15,18,27]. Tarantello [27] proved that if

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$b \geq \lambda_1(\lambda_1 - c^2)$, then (1.2) has a nontrivial solution. Lazer and McKenna [15] discussed the case $n = 1$ when $\Sigma$ is an interval and found that (1.2) has at least $2k - 1$ nontrivial solutions if $b > \lambda_k(\lambda_k - c^2)$. Micheletti and Pistoia [18] studied (1.2) for a more general nonlinearity which is "like" $b[(u + 1)^+ - 1]$ and obtained the multiplicity existence of nontrivial solutions. When $c^2 \geq \lambda_1$, Micheletti and Pistoia considered (1.2) when $b[(u + 1)^+ - 1]$ is replaced by a more general nonlinearity in [17], where they showed the existence of a nontrivial solution. Then Micheletti, Pistoia and Saccon [19] improved the result in [17] and gave the multiplicity results for the same problem.

In other papers [32–34] this kind of problem was also investigated under the Navier boundary condition. More recently, Ferrero and Gazzola [5] suggested one should consider the boundaries of a plate $\Omega = (0, \pi) \times (-l, l)$ which represents the roadway of a suspension bridge as follows. Because the edges $x = 0, \pi$ connect with the ground, they are assumed to be hinged and then

$$u(0, y) = u_{xx}(0, y) = u(\pi, y) = u_{xx}(\pi, y) = 0, \quad y \in (-l, l), \tag{1.3}$$

while the edges $y = \pm l$ are free and the boundary conditions there become (see [30, 2.40])

$$u_{yy}(x, \pm l) + \sigma u_{xx}(x, \pm l) = 0, \qquad u_{yyy}(x, \pm l) + (2 - \sigma)u_{xxy}(x, \pm l) = 0, \quad x \in (0, \pi). \tag{1.4}$$

The free boundaries (1.4) yield small stretching energy for the plate, so Ferrero and Gazzola took $c = 0$ in (1.1) and introduced a model for the stationary suspension bridge

$$\Delta^2 u + h(x, y, u) = f(x, y), \quad \text{in } \Omega \tag{1.5}$$

as well as a model for the nonlinear dynamical suspension bridge

$$u_{tt} + \Delta^2 u + \mu u_t + h(x, y, u) = f(x, y, t), \quad \text{in } \Omega \times (0, T), \tag{1.6}$$

here $h(x, y, u)$ is restoring force due to the hangers of the suspension bridge, $f(x, y)$ or $f(x, y, t)$ is the external force including the gravity.

Given an open rectangular plate $\Omega = (0, \pi) \times (-l, l) \subset \mathbb{R}^2$, we consider the following initial value problem

$$\begin{cases} u_{tt} + \Delta^2 u + \mu u_t + au = |u|^{p-2}u, & (x, y, t) \in \Omega \times [0, T], \\ u(x, y, 0) = u_0(x, y), & (x, y) \in \Omega, \\ u_t(x, y, 0) = u_1(x, y), & (x, y) \in \Omega, \end{cases} \tag{1.7}$$

with the boundary condition

$$\begin{cases} u(0, y, t) = u_{xx}(0, y, t) = u(\pi, y, t) = u_{xx}(\pi, y, t) = 0, & y \in (-l, l), \\ u_{yy}(x, \pm l, t) + \sigma u_{xx}(x, \pm l, t) = u_{yyy}(x, \pm l, t) + (2 - \sigma)u_{xxy}(x, \pm l, t) = 0, & x \in (0, \pi), \end{cases} \tag{1.8}$$

for every $t \in [0, T]$, where $T > 0$, $\mu > 0$, $2 < p < \infty$, $\sigma \in (0, \frac{1}{2})$ and $a = a(x, y, t)$ is a sign-changing and bounded measurable function. The initial data $u_0$, $u_1$ belong to suitable spaces, which will be specified later on.

Here we study the fourth order wave problem (1.7) with the boundary condition (1.8), because it comes from the physical model for the dynamic suspension bridge if we assume that $au$ describes the restoring force because of the hangers of the suspension bridge and $|u|^{p-2}u$ is the strong source term which represents the other external forces acting on the bridge. We explain this model briefly below, see for more details [5].