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A sufficient condition for the existence of a principal eigenvalue for nonlocal diffusion equations with applications

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ABSTRACT

Considerable work has gone into studying the properties of nonlocal diffusion equations. The existence of a principal eigenvalue has been a significant portion of this work. While there are good results for the existence of a principal eigenvalue equations on a bounded domain, few results exist for unbounded domains. On bounded domains, the Krein–Rutman theorem on Banach spaces is a common tool for showing existence. This article shows that generalized Krein–Rutman can be used on unbounded domains and that the theory of positive operators can serve as a powerful tool in the analysis of nonlocal diffusion equations. In particular, a useful sufficient condition for the existence of a principal eigenvalue is given.

1. Introduction

Nonlocal diffusion equations come up in a number of contexts. These range from biology [1,7,10,15] to materials science [3] and to graph theory. Much of the analysis to this point has been done on bounded domains [4]. The results presented here represent a first step in analyzing nonlocal diffusion equations on unbounded domains. The theory of positive operators on Banach lattices is employed to give useful conditions to show existence of a principal eigenvalue in the absence of compact domains. Many more results are likely to come from applying known results about positive operators to nonlocal diffusion equations. In particular, in Section 3.1, we use the existence of a particular type of maximum principle implies exponential convergence to zero.

The existence of a principal eigenvalue has been used to study many nonlocal diffusion equations, both linear and nonlinear [1,6]. The principal eigenvalue of an operator L is defined to be the eigenvalue with the greatest real part. For linear differential equations, the principal eigenvalue defines the long-run exponential dynamics of solutions. Even in the nonlinear case, such as in [1], the principal eigenvalue can be used for demonstrating stability. Energy methods and Fourier analysis have been used to provide polynomial bounds on the decay of nonlocal diffusion equations [2,8], with exponential decay shown in some special cases [8]. However, estimates of the principal eigenvalue would be able to more easily show exponential decay.

While much work has been done on the existence of principal eigenvalues on bounded domains, few general results exist for unbounded domains [4,9]. In [4], the existence of a principal eigenvalue was found

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for general domains but only for a specific type of kernel. Similarly, [9] shows upper and lower bounds for the principal eigenvalue but only for a specific form of kernel.

It is important to note that, even on bounded domains, existence of a principal eigenvalue is still not guaranteed. See, for example, the counterexample in Section 3.2. The work here uses the theory of positive operators to find a sufficient condition for the existence of a principal eigenvalue without any assumptions on the boundary conditions. The method also gives a technique for estimating the value of the eigenvalue. Further work is needed to characterize the multiplicity of the eigenvalue or the existence of a spectral gap.

One of the most common tools for analyzing nonlocal diffusion equations is the Krein–Rutman theorem. The theorem is stated in [14] as:

> Let E be an ordered real Banach space with total positive cone C, and let u be a compact positive endomorphism of E. If u has a spectral radius $r(u) > 0$, then $r(u)$ is a pole of the resolvent of maximal order on the spectral circle, with an eigenvector in C. A corresponding result holds for the adjoint u' in E'.

notation from Schaefer. The most stringent condition for using the theorem is that the operator is compact. For nonlocal diffusion equations defined on a bounded domain, it is usually simple to prove compactness. However, such operators are generally not compact on unbounded domains. The results here provide new tools for analyzing such operators.

The general equation of interest is:

$$\dot{u}(x,t) = \int_{\Omega} J(x,y)u(y,t)\,\mathrm{d}y - a(x)u(x,t) \tag{1.1}$$

where Ω is some connected, possibly unbounded subset of the real line $\mathbb{R}$ and $u \in L^p(\Omega)$ for $1 \leqslant p < \infty$. This equation is often interpreted as modeling some form population dispersal. Individuals propagate from point y to point x at a rate $J(x,y)$ and individuals die off at a rate $a(x)$ depending on their location. Note that the proofs here immediately generalizes to $\mathbb{R}^n$. a is assumed to be continuous and bounded in the sense that there exist c, c' such that:

$$0 < c < a(x) < c' \tag{1.2}$$

The two additional hypotheses on J are that $J(x,y) \geqslant 0$ and J is bounded with non-zero spectrum, where J:

$$\mathrm{J}u = \int_{\Omega} J(x,y)u(y)\,\mathrm{d}y \tag{1.3}$$

Define L to be the operator on the right hand side of (1.1):

$$\mathrm{L}u = \mathrm{J}u - au \tag{1.4}$$

The result in Theorem 1 gives a quite general condition for the existence of a principal eigenvalue for L. There is no particular reason that J has to be of integral type, but the discussion here will be limited to that case. If J is a more general positive operator, an appropriate compact topological space will have to be found to ensure the conclusion from Lemma 2. Several applications of the theorem are given in Section 3.

Define the two related linear operators:

$$\mathrm{L}u = \mathrm{J}u - au \qquad \mathrm{A}_\lambda u = \frac{\mathrm{J}u}{\lambda + a} \tag{1.5}$$