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Existence and multiplicity of weak solutions for elliptic Dirichlet problems with variable exponent

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A R T I C L E I N F O

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A B S T R A C T

Under appropriate growth conditions on the nonlinearity, the existence of multiple solutions for nonlinear elliptic Dirichlet problems with variable exponent is established. The approach is based on variational methods. Some applications and examples illustrate the obtained results.

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1. Introduction

The aim of this paper is to investigate the following Dirichlet problem

$$\begin{cases} -\Delta_{p(x)} u = \lambda f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (D_{\lambda,f})$$

where $\Delta_{p(x)} u := div(|\nabla u|^{p(x)-2}\nabla u)$ denotes the $p(\cdot)$-Laplace operator, $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary, $p \in C(\bar{\Omega})$ is a function regular enough (see Section 2) satisfying

$$1 < p^- := \inf_{x\in\Omega} p(x) \le p(x) \le p^+ := \sup_{x\in\Omega} p(x) < +\infty, \qquad (1)$$

$f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and λ is a real positive parameter. The study of differential equations and variational problems involving operators with variable exponents growth conditions has received more and more interest in the last few years. It was found that these non-standard variational problems are related to modeling of so-called electrorheological fluids. Moreover, progress in physics has made the study of fluid mechanical properties of these fluids an important issue (see for instance [11,21,24]).

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The problem $(D_{\lambda,f})$ has been studied by several authors (see for instance [3–5,10,9,13,12,14,15,8,18,19, 19,20,22,26]). In particular, we cite the papers [4,15,26,20,8,22] in which the existence of one or three weak solutions for the problem $(D_{\lambda,f})$ has been established. We start from the following condition:

(f_1) there exist $a_1, a_2 \in [0, +\infty[$ and $q \in C(\bar{\Omega})$ with $1 < q(x) < p^*(x)$ for each $x \in \bar{\Omega}$, such that

$$\left|f(x,t)\right| \le a_1 + a_2|t|^{q(x)-1}$$

for each $(x,t) \in \Omega \times \mathbb{R}$, where

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N \\ \infty & \text{if } p(x) \ge N. \end{cases}$$

We observe that

- in [15,26] the existence of one nontrivial weak solution for the problem $(D_{\lambda,f})$ (case $\lambda = 1$) has been established assuming, among others, that the nonlinear term f verifies the (f_1) with $q^- > p^+$;
- in [4] three weak solutions for the problem $(D_{\lambda,f})$ are obtained when $p^- > N$ and f verifies the following less general growth condition

$$\left|f(x,t)\right| \le c\left(1 + |t|^{s-1}\right) \quad \forall (x,t) \in \Omega \times \mathbb{R}$$

  with $s \in [1, p^-[$;
- in [20,8,22] the existence of one nontrivial weak solution for the problem $(D_{\lambda,f})$ has been investigated for functions f satisfying (f_1) with $q(x) \le p^*(x)$ and $\{q(x) = p^*(x)\} \neq \emptyset$.

Here, we deal with the problem $(D_{\lambda,f})$ when the nonlinearity f has the subcritical growth (f_1) and, via variational methods, we obtain the existence of at least one, two or three weak solutions whenever the parameter λ belongs to a precise positive interval. The main tools are critical points theorems established in [1,2,6].

Here, as an example, we present a special case of one of our results:

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (f_1). Moreover, assume that*

(f_2')

$$\lim_{t \to 0^+} \frac{f(t)}{t^{p^- -1}} = +\infty.$$

Then, there exists $\lambda^* > 0$ such that for each $\lambda \in]0, \lambda^*[$, the problem

$$\begin{cases} -\Delta_{p(x)} u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

admits at least one nontrivial weak solution.

The paper is organized as follows. In Section 2 we recall some properties of variable exponent spaces and in Section 3 the existence of one weak solution for the problem $(D_{\lambda,f})$ is obtained. In Section 4, we apply one of the main tools to establish the existence of two distinct weak solutions for problem $(D_{\lambda,f})$. Finally, in Section 5, the existence of three weak solutions for the problem $(D_{\lambda,f})$ is achieved.

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