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Quasi-banded operators, convolutions with almost periodic or quasi-continuous data, and their approximations

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ABSTRACT

We study the stability and Fredholm property of the finite sections of quasi-banded operators acting on L^p spaces over the real line. This family is significantly larger than the set of band-dominated operators, but still permits to derive criteria for the stability and results on the splitting property, as well as an index formula in the form as it is known for the classical cases. In particular, this class covers convolution type operators with semi-almost periodic and quasi-continuous symbols, and operators of multiplication by slowly oscillating, almost periodic or even more general coefficients.

1. Introduction

Let X be a separable and infinite dimensional Banach space, I the identity operator on X, $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on X, and $\mathcal{K}(X)$ the ideal of the compact operators in $\mathcal{L}(X)$.

For $A \in \mathcal{L}(X)$, the basic problem of (linear) numerical mathematics is to provide approximations to the solution of the operator equation

$$Au = v \quad \text{for given } v \in X. \tag{1}$$

The standard procedure is to choose a sequence of projections P_n which converge strongly to the identity operator, and a sequence of operators $A_n : \operatorname{im} P_n \to \operatorname{im} P_n$ such that the $A_n P_n$ converge strongly to A, and to replace Eq. (1) by the "simpler" equations

$$A_n u_n = P_n v \quad \text{for } n = 1, 2, \ldots, \tag{2}$$

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the solutions u_n being sought in $\operatorname{im} P_n$. The crucial question is whether this method *applies*, i.e. whether Eqs. (2) possess unique solutions u_n for every right-hand side v and for every sufficiently large n, say for $n \geq n_0$, and whether the sequence $(u_n)_{n\geq n_0}$ converges to the solution u of the original equation (1). The applicability of the method is equivalent to the stability of the operator sequence (A_n), i.e. to the invertibility of the operators A_n for n large enough and to the uniform boundedness of the norms of their inverses.

The need for the strong convergence of projections stems from the close relationship between strong convergence and the ideal of compact operators, which is at the heart of most proofs. In this context, the observation that the multiplication of a strongly convergent sequence by a compact operator gives uniform convergence plays a fundamental role.

A particular projection family with the strong convergence property is given by the so-called finite section projections, defined as the operators $P_n = \chi_{[-n,n]}I$ of multiplication by the characteristic functions of the interval $[-n, n]$. The first results regarding this family, due to Gohberg and Feldman [6], applied to some classes of continuous symbol convolution operators in $L^p(\mathbb{R})$. In 1984, Böttcher [2] gave conditions for the applicability of the method to Wiener–Hopf operators in $L^p(\mathbb{R}^+)$ with piecewise continuous symbols. S. Roch, B. Silbermann, A. Karlovich and two of the authors gave later several contributions, extending the previous results in various directions (Hankel operators, other types of symbols, operators belonging to algebras generated by the canonical cases, see [20–23,13]).

In order to tackle non-strongly convergent projections, as it is the case for the finite section projections in $l^\infty(\mathbb{Z})$ or $L^\infty(\mathbb{R})$, S. Roch and B. Silbermann [24] (also [17, 4.36 et seq.]) advanced the idea of changing the above mentioned connection between compactness and strong convergence to a definition. That is, starting with the projection sequence $\mathcal{P} := (P_n)_{n\in\mathbb{N}}$, to substitute the usual compact operators by compact-like operators related to that sequence. The $\mathcal{P}$-compact operators will be exactly those operators $K \in \mathcal{L}(X)$ such that $\|KP_n - K\|$ and $\|P_nK - K\|$ tend to zero as $n \to \infty$.

This insight changed the whole context of the problem. It is now possible to define a subalgebra $\mathcal{L}(X, \mathcal{P}) \subset \mathcal{L}(X)$ where the $\mathcal{P}$-compact operators act as an ideal, to naturally define $\mathcal{P}$-convergence and $\mathcal{P}$-Fredholmness and to obtain conditions for the stability of approximating sequences for many classes of operators (see [19,14,29,28]).

The present paper continues to explore this new context, applies and extends the theory in several directions. In particular, this paper gives the following contributions.

Together with the stability of operator sequences we study a more general Fredholm theory in sequence algebras, including the splitting property and an index formula in the case of $\mathcal{P}$-strongly converging finite section methods. This setting particularly covers convolution type operators in $L^p(\mathbb{R})$, with $p \in [1, \infty]$.

We introduce the class $\mathcal{Q}$ of quasi-banded operators, extending the previously known notion of band-dominated operators. In particular, we obtain results on its inverse- and Fredholm closedness, and results on the stability of the finite section method for such operators. This new context allows for the treatment of the finite section method for convolution type operators with symbols in a wide variety of function spaces, including slowly oscillating, almost periodic, semi-almost periodic and even general quasi-continuous functions. It is also possible to combine the convolution operators with multiplication operators by functions of again a variety of spaces (including piecewise continuous, bounded and uniformly continuous, slowly oscillating, and almost-periodic) and to obtain relatively simple conditions for stability. Note that the results cover the cases $p \in \{1, \infty\}$, in which all Fourier convolution operators are contained in $\mathcal{Q}$.

By using a special transformation, we are also able to include the flip J, defined by $(Ju)(x) := u(-x)$. This allows to apply our results also for Wiener–Hopf plus Hankel operators, for example. Furthermore, it is shown that, for symbol functions living in a large class that includes semi-almost periodic functions, the respective Hankel operator is $\mathcal{P}$-compact.

An interesting observation is that the general results obtained by this method do not need the use of local principles. Without local principles, it is possible to cover convolution operators whose symbol has a jump at infinity. We also show that it is not possible to apply this simple method if the symbol function has