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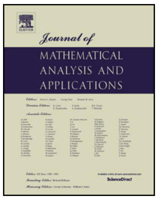

Nonlinear generalized functions and jump conditions for a standard one pressure liquid–gas model

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ABSTRACT

The mixture of a liquid and a gas is classically represented by one pressure models. These models are a system of PDEs in nonconservative form and shock wave solutions do not make sense within the theory of distributions: they give rise to products of distributions that are not defined within distribution theory. But they make sense by applying a theory of nonlinear generalized functions to these equations. In contrast to the familiar case of conservative systems the jump conditions cannot be calculated a priori. Jump conditions for these nonconservative systems can be obtained using the theory of nonlinear generalized functions by inserting some adequate physical information into the equations. The physical information that we propose to insert for the one pressure models of a mixture of a liquid and a gas is a natural mathematical expression in the theory of nonlinear generalized functions of the fact that liquids are practically incompressible while gases are very compressible, and so they do not satisfy equally well their respective state laws on the shock waves. This modelization gives well defined explicit jump conditions. The great numerical difficulty for solving numerically nonconservative systems is due to the fact that slightly different numerical schemes can give significantly different results. The jump conditions obtained above permit to select the numerical schemes and validate those that give numerical solutions that satisfy these jump conditions, which can be an important piece of information in the absence of other explicit discontinuous solutions and of precise observational results. We expose with care the mathematical originality of the theory of nonlinear generalized functions (an original abstract analysis issued by the Leopoldo Nachbin team on infinite dimensional holomorphy) that permits to state mathematically physical facts that cannot be formulated within distribution theory, and are the key for the removal of “ambiguities” that classically appear when one tries to calculate on “multiplications of distributions” that occur in the differential equations of physics.

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1. Introduction

In the isothermal case and some other simplifications (no gravitational effects and no transfer of momentum between the two fluids) the standard one pressure model used to describe mathematically a mixture of two immiscible fluids [29, pp. 372, 373], [30, p. 286], [20,21,27] dropping the interface term, and references there, is stated as

$$(\rho_l\alpha_l)_t + (\rho_l\alpha_l u_l)_x = 0, \tag{1}$$

$$(\rho_g\alpha_g)_t + (\rho_g\alpha_g u_g)_x = 0, \tag{2}$$

$$(\rho_l\alpha_l u_l)_t + \left(\rho_l\alpha_l(u_l)^2\right)_x + \alpha_l p_x = 0, \tag{3}$$

$$(\rho_g\alpha_g u_g)_t + \left(\rho_g\alpha_g(u_g)^2\right)_x + \alpha_g p_x = 0, \tag{4}$$

$$\alpha_l + \alpha_g = 1, \tag{5}$$

$$p = \mathcal{P}_l(\rho_l), \qquad p = \mathcal{P}_g(\rho_g), \tag{6}$$

where the two fluids are denoted by the indices l and g respectively for “liquid” and “gas” since we will consider only this particular case in the sequel (mixture of oil and natural gas in extraction tubes of oil exploitation, mixture of hot water and vapor of water in cooling tubes of nuclear power stations). The physical variables are $(i = l, g)$

$\rho_i(x,t) =$ density of fluid i,
$u_i(x,t) =$ velocity of fluid i,
$\alpha_i(x,t) =$ volumic proportion of fluid i,
$p(x,t) =$ pressure in the mixture,
$\mathcal{P}_i =$ state law of fluid i.

Eqs. (1), (2) are the continuity equations for each fluid: they express mass conservation for each fluid. Eqs. (3) and (4) are the Euler equations for each fluid: they express momentum conservation for each fluid. In oil exploitation they are completed by a term involving gravitation and by a term corresponding to possible momentum transfer between the two phases and possibly by a viscous term, see [27]: we will not take these terms into account. Eqs. (6) are the two state laws of the liquid and the gas in the isothermal case. In the case of variable temperature the state laws involve the energy and velocity

$$p = \mathcal{P}_i(\rho_i, e_i, u_i) \tag{7}$$

so that the two energy equations cannot be dissociated from the other equations (1)–(4). These energy equations are [27,29]

$$(\rho_l\alpha_l e_l)_t + (\rho_l\alpha_l u_l e_l)_x + (\alpha_l p u_l)_x + p.(\alpha_l)_t = 0, \tag{8}$$

$$(\rho_g\alpha_g e_g)_t + (\rho_g\alpha_g u_g e_g)_x + (\alpha_g p u_g)_x + p.(\alpha_g)_t = 0, \tag{9}$$

where $e_i(x,t)$ is the density of total energy of fluid i.

The system is nonconservative, i.e. the derivatives cannot be transferred to test functions because of the terms $\alpha_i\frac{\partial p}{\partial x}$ in (3), (4) and $p\frac{\partial \alpha_i}{\partial t}$ in (8), (9). In the case of shock waves the variables p and α_i are discontinuous and these terms appear as undefined products of the form of the Heaviside function multiplied by the Dirac delta distribution. Not only these products do not make sense mathematically but also one cannot obtain formulas for the jump conditions since formal calculations give contradictory results depending on the way they are done. Correspondingly, numerical methods become ambiguous in that it has been observed that slight modifications in the formulas of the scheme can give very different results [13, p. 47], which can be explained in the nonlinear theory of generalized functions by the existence of different Heaviside functions.