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Regularity for the fractional Gelfand problem up to dimension 7

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A R T I C L E I N F O

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We study the problem $(-\Delta)^s u = \lambda e^u$ in a bounded domain $\Omega \subset \mathbb{R}^n$, where λ is a positive parameter. More precisely, we study the regularity of the extremal solution to this problem. Our main result yields the boundedness of the extremal solution in dimensions $n \le 7$ for all $s \in (0,1)$ whenever Ω is, for every $i = 1, ..., n$, convex in the x_i-direction and symmetric with respect to $\{x_i = 0\}$. The same holds if $n = 8$ and $s \gtrsim 0.28206...$, or if $n = 9$ and $s \gtrsim 0.63237...$. These results are new even in the unit ball $\Omega = B_1$.

1. Introduction and results

Let $s \in (0,1)$ and Ω be a bounded smooth domain in $\mathbb{R}^n$, and consider the problem

$$\begin{cases} (-\Delta)^s u = \lambda e^u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.1}$$

Here, λ is a positive parameter and $(-\Delta)^s$ is the fractional Laplacian, defined by

$$(-\Delta)^s u(x) = c_{n,s} \mathrm{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy. \tag{1.2}$$

The aim of this paper is to study the regularity of the so-called extremal solution of the problem (1.1).

For the Laplacian $-\Delta$ (which corresponds to $s = 1$) this problem is frequently called the Gelfand problem [16], and the existence and regularity properties of its solutions are by now quite well understood [19,17,21,20,9]; see also [15,22].

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Indeed, when $s = 1$ one can show that there exists a finite extremal parameter λ^* such that if $0 < \lambda < \lambda^*$ then it admits a minimal classical solution u_λ, while for $\lambda > \lambda^*$ it has no weak solution. Moreover, the pointwise limit $u^* = \lim_{\lambda\uparrow\lambda^*} u_\lambda$ is a weak solution of problem with $\lambda = \lambda^*$. It is called the extremal solution. All the solutions u_λ and u^* are stable solutions.

On the other hand, the existence of other solutions for $\lambda < \lambda^*$ is a more delicate question, which depends strongly on the regularity of the extremal solution u^*. More precisely, it depends on the boundedness of u^*.

It turns out that the extremal solution u^* is bounded in dimensions $n \le 9$ for any domain Ω [20,9], while $u^*(x) = \log \frac{1}{|x|^2}$ is the (singular) extremal solution in the unit ball when $n \ge 10$. This result strongly relies on the stability of u^*. In the case $\Omega = B_1$, the classification of all radial solutions to this problem was done in [19] for $n = 2$, and in [17,21] for $n \ge 3$.

For more general nonlinearities $f(u)$ the regularity of extremal solutions is only well understood when $\Omega = B_1$. As in the exponential case, all extremal solutions are bounded in dimensions $n \le 9$, and may be singular if $n \ge 10$ [6]. For general domains Ω the problem is still not completely understood, and the best result in that direction states that all extremal solutions are bounded in dimensions $n \le 4$ [5,25]. In domains of double revolution, all extremal solutions are bounded in dimensions $n \le 7$ [7]. For more information on this problem, see [3] and the monograph [14].

For the fractional Laplacian, the problem was studied by J. Serra and the author [24] for general nonlinearities f. We showed that there exists a parameter λ^* such that for $0 < \lambda < \lambda^*$ there is a branch of minimal solutions u_λ, for $\lambda > \lambda^*$ there is no bounded solutions, and for $\lambda = \lambda^*$ one has the extremal solution u^*, which is a stable solution. Moreover, depending on the nonlinearity f and on n and s, we obtained L^∞ and H^s estimates for the extremal solution in general domains Ω. Note that, as in the case $s = 1$, once we know that u^* is bounded then it follows that it is a classical solution; see for example [23].

For the exponential nonlinearity $f(u) = e^u$, our results in [24] yield the boundedness of the extremal solution in dimensions $n < 10s$. Although this result is optimal as $s \to 1$, it is not optimal, however, for smaller values of $s \in (0, 1)$. More precisely, an argument in [24] suggested the possibility that the extremal solution u^* could be bounded in all dimensions $n \le 7$ and for all $s \in (0, 1)$. However, our results in [24] did not give any L^∞ estimate uniform in s.

The aim of this paper is to obtain better L^∞ estimates for the fractional Gelfand problem (1.1) whenever Ω is even and convex with respect to each coordinate axis. Our main result, stated next, establishes the boundedness of the extremal solution u^* whenever (1.3) holds and, in particular, whenever $n \le 7$ independently of $s \in (0, 1)$. As explained in Remark 2.2, we expect this result to be optimal.

Theorem 1.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^n$ which is, for every $i = 1, ..., n$, convex in the x_i-direction and symmetric with respect to $\{x_i = 0\}$. Let $s \in (0, 1)$, and let u^* be the extremal solution of problem (1.1). Assume that either $n \le 2s$, or that $n > 2s$ and*

$$\frac{\Gamma(\frac{n}{2})\Gamma(1+s)}{\Gamma(\frac{n-2s}{2})} > \frac{\Gamma^2(\frac{n+2s}{4})}{\Gamma^2(\frac{n-2s}{4})}. \tag{1.3}$$

Then, u^* is bounded. In particular, the extremal solution u^* is bounded for all $s \in (0, 1)$ whenever $n \le 7$. The same holds if $n = 8$ and $s \gtrsim 0.28206...$, or if $n = 9$ and $s \gtrsim 0.63237...$.

The result is new even in the unit ball $\Omega = B_1$.

We point out that, for $n = 10$ condition (1.3) is equivalent to $s > 1$.

Let us next comment on some works related to problem (1.1).

On the one hand, for the power nonlinearity $f(u) = (1+u)^p$, $p > 1$, the problem has been recently studied by Dávila, Dupaigne, and Wei [13]. Their powerful methods are based on a monotonicity formula and a blow-up argument, using the ideas introduced in [12] to study the case of the bilaplacian, $s = 2$. For this case $s = 2$, extremal solutions with exponential nonlinearity have been also studied; see for example [10].