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Local time of self-affine sets of Brownian motion type and the jigsaw puzzle problem

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ABSTRACT

Let $\Omega \subset [0,1] \times [0,1]$ be the solution of the set equation:

$$\Omega = \bigcup_{i=1}^{k} (\varphi_{I_i,1} \times \varphi_{J_i,\tau_i})(\Omega),$$

where for an interval $I = [a,b] \subset [0,1]$ and $\tau \in \{-1,1\}$, $\varphi_{I,\tau} : [0,1] \to I$ is the linear map such that $\varphi_{I,1}(0) = a$, $\varphi_{I,1}(1) = b$, $\varphi_{I,-1}(0) = b$, $\varphi_{I,-1}(1) = a$, and $\{I_i;\ i = 1,\cdots,k\}$ is a partition of $[0,1]$ with $|J_i| = |I_i|^{1/2}$. Thus, Ω is a graph of a Borel function f_Ω almost surely and it is called a self-affine set of Brownian motion type. Let λ be the Lebesgue measure on $[0,1]$ and let $\mu_\Omega = \lambda \circ f_\Omega^{-1}$. The density $\rho_\Omega = \frac{d\mu_\Omega}{d\lambda}$, if it exists, is called the local time of Ω and it has been studied. It is known that $\dim_H \Omega = 3/2$ if ρ_Ω exists. In the present study, ρ_Ω is obtained by solving the so-called jigsaw puzzle on $\{J_i, \tau_i;\ i = 1,\cdots,k\}$, i.e., the problem of decomposing ρ_Ω into a sum of its self-similar images with the support J_i and the orientation τ_i for $i = 1,\cdots,k$.

1. Introduction

Assume that

$$(\#1)\qquad \begin{aligned} &k \text{ is an integer where } k \geq 2,\ 0 = s_0 < s_1 < \cdots < s_k = 1, \text{ and} \\ &I_i = [s_{i-1}, s_i]\ (i = 1,\cdots,k) \text{ are intervals.} \end{aligned}$$

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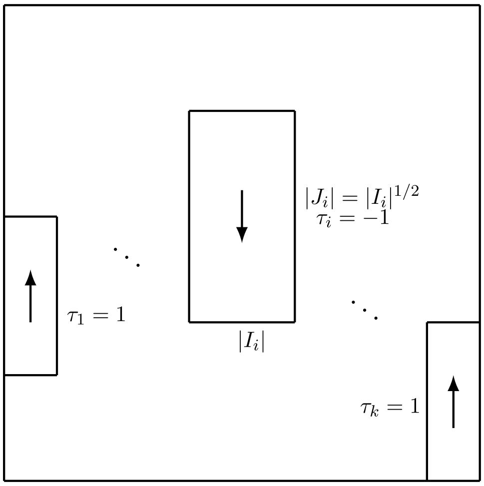

Fig. 1. Set equation for Ω, where $\uparrow$ and $\downarrow$ correspond to the upward or downward linear maps, respectively, from the unit square to $I_i \times J_i$ $(i = 1, 2, \cdots, k)$.

In addition, assume further

$$(\#2) \quad \begin{array}{l} J_i \ (i = 1, \cdots, k) \text{ are closed intervals in } [0,1] \text{ such that } |J_i| = |I_i|^{1/2} \ (i = 1, \cdots, k), \\ \text{and } \tau_i \in \{-1, 1\} \ (i = 1, \cdots, k), \end{array}$$

where $|I|$ denotes the length $b - a$ of the interval $I = [a, b]$.

For an interval $I = [a, b]$ where $0 \le a < b \le 1$ and $\tau \in \{-1, 1\}$, we define a linear map $\varphi_{I,\tau} : [0, 1] \to I$ by

$$\varphi_{I,\tau}(x) = \begin{cases} a(1-x) + bx & (\tau = 1), \\ b(1-x) + ax & (\tau = -1). \end{cases}$$

We may denote $\varphi_{I,1}$ simply by φ_I. Therefore, $\varphi_I \times \varphi_{J,\tau}$ with intervals $I = [a, b]$, $J = [c, d]$ in $[0, 1]$ is the linear bijection $[0, 1] \times [0, 1] \to I \times J$ such that $\varphi_I \times \varphi_{J,1}$ maintains the orientation, whereas $\varphi_I \times \varphi_{J,-1}$ reverses the vertical orientation. The former map is called the upward map and the latter map is called the downward map, which are denoted by upward and downward arrows, respectively. See Fig. 1.

Definition 1. Let $\Omega = \Omega(I_1, \cdots, I_k; J_1, \cdots, J_k; \tau_1, \cdots, \tau_k)$ be the compact set in $[0, 1] \times [0, 1]$ that satisfies the following set equation [3,5]:

$$\Omega = \bigcup_{i=1}^{k} (\varphi_{I_i} \times \varphi_{J_i,\tau_i})(\Omega). \tag{1.1}$$

We refer to Ω as a self-affine set of *Brownian motion type*.

Self-affine sets of Brownian motion type were studied in [1,8], and [7] from the perspective of stochastic or functional analysis. McMullen's notion of general Sierpiński carpets [9] is similar to our notion, but there are some differences, i.e., our notion has more freedom to choose the position and the orientation of J_i, whereas $\log |J_i| / \log |I_i|$ is not necessarily $1/2$ in McMullen's notion and there can be more than one rectangle on the same vertical line. In addition, continuous self-affine functions with $|I_i| = 1/k$ $(i = 1, \cdots, k)$ were studied previously [6], where it was proved that functions with different coprime k and k' that coincide are linear functions.