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Spectral properties of the operator of Bessel potential type ☆

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ABSTRACT

Let Ω be a convex bounded domain in $\mathbf{R}^m$ having regular boundary. In this paper, we study integral operators B_m^α on $L^2(\Omega)$ of Bessel potential type. If $N(\lambda)$ denotes the number of eigenvalues of B_m^α that are $\geq \lambda$, for $\lambda > 0$, we find the asymptotics of the regularized eigenvalue distribution function $\lambda \mapsto \int_\lambda^\infty N(\mu)d\mu$ when $\lambda \to 0^+$. As a consequence, we find the regularized traces of these operators.

1. Introduction

We study the operators

$$B_m^\alpha : L^2(\Omega) \longrightarrow L^2(\Omega), \quad \Omega \subset \mathbb{R}^m$$

defined by

$$B_m^\alpha f(x) = \int_\Omega G_\alpha(x-y) f(y) dy$$

where

$$G_\alpha(x) = \frac{2^{\frac{2-m-\alpha}{2}}}{\pi^{\frac{m}{2}} \Gamma(\frac{\alpha}{2})} \cdot \frac{K_{\frac{m-\alpha}{2}}(|x|)}{|x|^{\frac{m-\alpha}{2}}}, \quad \alpha > 0.$$

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Here $x = (x_1, x_2, \dots, x_m)$, $|x| = \sqrt{\sum_{i=1}^m x_i^2}$, $dx = dx_1 dx_2 \dots dx_m$ and K_ν is the McDonald function:

$$K_\nu(z) = \frac{\pi}{2\sin\nu\pi}\big(I_{-\nu}(z) - I_\nu(z)\big), \quad \nu \notin \mathbb{Z},$$
$$K_n(z) = \lim_{\nu\to n} K_\nu(z), \quad n = 0, \pm 1, \pm 2 \dots$$

and

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(\frac{z}{2})^{\nu+2k}}{k!\Gamma(\nu+k+1)}.$$

The domain Ω is convex, bounded with sufficiently regular boundary. By $|\Omega|$ we will denote the Lebesgue measure of Ω. We call the convolution operators with kernel G_α *Bessel potentials*. They occur in several places, including the theory of fractional integration, Operator Theory, Harmonic Analysis and Mathematical Physics.

If $\alpha > 0$, the operators B_m^α are compact on $L^2(\Omega)$. They are close to negative fractional power of the operator $I - \Delta$.

The operators $(-\Delta)^s$ appear in numerous fields (such as mathematical analysis, mathematical physics, mathematical biology, ...). Two term asymptotics expansion of the sum of eigenvalues of fractional Laplacian $(-\Delta)^s$, when $0 < s < 1$, has been found in the paper [9].

In [12], the author finds the two term Weyl type asymptotics for the eigenvalues of the one-dimensional fractional Laplace operator $(-\Delta)^s$ $(0 < s < 1)$ on the interval $(-1, 1)$.

In [1] it was proved that the second term in the asymptotic expansion of the trace of the semigroup of a symmetric stable process (fractional powers of the Laplacian) of order α, for $0 < \alpha < 2$, in a Lipschitz domain is given by the surface area of the boundary domain, as $t \to 0+$.

In [16] the two term asymptotic formula for the eigenvalue distribution function for the n-th power of the Laplacian (and also for more general operators) has been given.

In this paper we denote by $\widehat{f}$ and $\check{f}$ the direct and inverse Fourier transform:

$$\widehat{f}(x) = \int_{\mathbb{R}^m} e^{-ix\cdot y} f(y)dy$$
$$\check{f}(x) = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} e^{ix\cdot y} f(y)dy$$

(with $x \cdot y$, $x, y \in \mathbb{R}^m$, denoting the inner product in $\mathbb{R}^m$).

It is well known (see, e.g., [17]) that G_α is a positive function and

$$\widehat{G}_\alpha(x) = \frac{1}{(1+|x|^2)^{\frac{\alpha}{2}}},$$
$$\int_{\mathbb{R}^m} G_\alpha(x)dx = 1.$$

It follows that the operator

$$f \mapsto \int_{\mathbb{R}^m} G_\alpha(x-y) f(y)dy$$