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A class of minimization problems related to quasilinear equations

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ABSTRACT

For a class of minimization problems related to quasilinear equations we establish the existence and nonexistence of minimizers by a minimization argument and the Pohozaev type identity.

1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ with $N \geq 3$. We are concerned with attainability of the quasilinear minimization problem in $H_0^1(\Omega)$

$$S(a,b,q;\Omega) = \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{L^q(\Omega)}=1}} \int_\Omega \left(1 + |x|^a |u|^b\right) |\nabla u|^2 \, dx. \tag{1.1}$$

In this paper, we consider that $0 \leq a < N$, $b \geq 0$ and q is related to a and b. A typical example is $a = 0$ and $b = 0$. It is well known that $S(0,0,q;\Omega)$ is achieved for $1 \leq q < \frac{2N}{N-2}$, and $S(0,0,\frac{2N}{N-2};\Omega)$ is never achieved unless $\Omega = \mathbb{R}^N$ (e.g., [14,15]). If $0 \notin \overline{\Omega}$, the problem is essentially equivalent to the case $a = 0$, thus from now on we will consider the case of $0 \in \Omega$. We assume without loss of generality that $u \geq 0$.

The minimization problem (1.1) is related to [1]. In [1], Bae et al. considered $a \geq 0$, $0 \leq b \leq \frac{2N}{N-2}$ and $q = \frac{2N}{N-2}$. They proved that the minimizer exists only in the range $a < \frac{N-2}{2}b$ which corresponds to a dominant nonlinear term. On the contrary, the linear term for $a \geq \frac{N-2}{2}b$ prevents the existence of the minimizer. They used scaling argument to study the problem and followed the method from [3]. In the celebrated paper [3], Brezis and Nirenberg considered the equation

$$\begin{cases} -\Delta u = u^{q-1} + \lambda u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

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where $q = \frac{2N}{N-2}$ is the critical exponent and λ is a real constant. They showed the equation has a solution $u \in H_0^1(\Omega)$ if $N \geq 4$ and $0 < \lambda < \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ is the principle eigenvalue of $-\Delta$ on Ω. The minimization problem corresponding to Eq. (1.2) is

$$\inf_{\substack{u\in H_0^1(\Omega)\\ \|u\|_{L^q(\Omega)}=1}} \int_\Omega \left(|\nabla u|^2 - \lambda u^2\right) dx. \tag{1.3}$$

Comparing the minimization problem (1.1) with the minimization problem (1.3), we find that they are both perturbations of the critical minimization problem

$$\inf_{\substack{u\in H_0^1(\Omega)\\ \|u\|_{L^q(\Omega)}=1}} \int_\Omega |\nabla u|^2 \, dx. \tag{1.4}$$

But the two problems have different motivations.

On the other hand, Wang et al. [13,9] considered the following quasilinear Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u - \Delta\left(|u|^{\frac{b}{2}+1}\right)|u|^{\frac{b}{2}-1}u = \lambda |u|^{q-2}u & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.5}$$

where $b > 0$ and $2 < q < (\frac{b}{2}+1)\frac{2N}{N-2}$. Quasilinear equation of form (1.5) appears more naturally in mathematical physics and has been derived as a model of several physical phenomena such as [7] and [8]. The minimization problem related to Eq. (1.5) is as follows:

$$\inf_{\substack{u\in X\\ \|u\|_{L^q(\mathbb{R}^N)}=1}} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + Vu^2 + |u|^b|\nabla u|^2\right) dx, \tag{1.6}$$

where $X = \{u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(x)u^2\,dx < \infty\}$ or $X = \{u \in H^1(\mathbb{R}^N) \mid u(x) = u(|x|)\}$ with norm given by $\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + Vu^2)\,dx$ depending on different conditions on V. The problem (1.6) is essentially related to our model (1.1). They established the existence of ground states of solutions by a minimization argument. After that, they reconsidered the same problem by using change of variables and Nehari manifold in [10] and [11]. In [4], Colin and Jeanjean also introduced a change of variable, and gave a shorter proof of the results of [10], which does not use Orlicz spaces, but rather is developed in $H^1(\mathbb{R}^N)$.

The minimization problem (1.1) is formally related to the following quasilinear elliptic equation

$$\begin{cases} -\mathrm{div}\left(\left(1+|x|^a|u|^b\right)\nabla u\right) + \dfrac{b}{2}|x|^a|u|^{b-2}u|\nabla u|^2 = \mu|u|^{q-2}u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where μ is the Lagrange multiplier. If $\frac{2N}{N-2} < q < (\frac{b}{2}+1)\frac{2N}{N+a-2}$, then q seems to be a supercritical exponent. In fact, we will prove that it is only a subcritical case for the new critical exponent $(\frac{b}{2}+1)\frac{2N}{N+a-2}$. For the new critical exponent, we will use the Pohozaev type identity to discuss nonexistence of solutions. In the case $a = \frac{N-2}{2}b$, we will get a little more results about nonexistence of solutions.

Our main result in this paper can be stated as follows:

Theorem 1.1. *Let $0 \in \Omega$ be a smooth bounded domain in $\mathbb{R}^N$ with $N \geq 3$. Consider the minimization problem (1.1) with $0 \leq a < N$ and $b \geq 0$. Then*

(i) *If $a \geq \frac{N-2}{2}b$, $S(a,b,q;\Omega)$ is achieved in $u \in H_0^1(\Omega) \cap L^q(\Omega)$ for $1 \leq q < \frac{2N}{N-2}$.*