J. Math. Anal. Appl. 419 (2014) 347–354

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Unbounded order convergence in dual spaces

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ARTICLE INFO

Article history:
Received 22 November 2013
Available online 2 May 2014
Submitted by M. Mathieu

Keywords:
Unbounded order convergence
Weak star convergence
Abstract martingales
Atomic Banach lattices
Positive Grothendick property
Dual positive Schur property

ABSTRACT

A net (x_α) in a vector lattice X is said to be unbounded order convergent (or uo-convergent, for short) to $x \in X$ if the net $(|x_\alpha - x| \wedge y)$ converges to 0 in order for all $y \in X_+$. In this paper, we study unbounded order convergence in dual spaces of Banach lattices. Let X be a Banach lattice. We prove that every norm bounded uo-convergent net in X^* is w^*-convergent iff X has order continuous norm, and that every w^*-convergent net in X^* is uo-convergent iff X is atomic with order continuous norm. We also characterize among σ-order complete Banach lattices the spaces in whose dual space every simultaneously uo- and w^*-convergent sequence converges weakly/in norm.

1. Introduction

This paper is a continuation of [4]. We follow the terminology and notations from [4]. Recall that a net (x_α) in a vector lattice X is ***unbounded order convergent*** (or uo-convergent, for short) to $x \in X$ if $|x_\alpha - x| \wedge y \xrightarrow{o} 0$ for all $y \in X_+$. In this case, we write $x_\alpha \xrightarrow{uo} x$. It is easily seen that a sequence (x_n) in $L_1(\mu)$ uo-converges to $x \in L_1(\mu)$ iff (x_n) converges to x almost everywhere. Let $\mathbb{R}^A$ be the vector lattice of all real-valued functions on a non-empty set A, equipped with the pointwise order. It is easily seen that a net (x_α) in $\mathbb{R}^A$ uo-converges to $x \in \mathbb{R}^A$ iff it converges pointwise to x.

The study of uo-convergence was initiated in [8,3]. In [10], Wickstead initiated the study of relations between uo-convergence and topological properties of the underlying spaces. He characterized the spaces in which uo-convergence of nets implies weak convergence and vice versa. In [4], Xanthos and the author studied nets which simultaneously have weak and uo convergence properties, and characterized among σ-order complete Banach lattices the spaces in which every weakly and uo-convergent sequence is norm convergent.

In this paper, we study uo-convergence in dual spaces. Let X be a Banach lattice. We are motivated by [10] and [4] to consider the following:

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http://dx.doi.org/10.1016/j.jmaa.2014.04.067

(1) characterize the spaces X such that in its dual space X^*, uo-convergence implies w^*-convergence and vice versa
(2) study nets/sequences in X^* which simultaneously have uo- and w^*-convergence properties, and characterize the spaces in whose dual space simultaneous uo- and w^*-convergence imply weak/norm convergence.

The remark at the end of this paper suggests the notion of uo-convergence in dual spaces could serve as a tool for the study of geometry of Banach lattices. We first remark here a few useful facts about uo-convergence.

Lemma 1.1. *Let X be a σ-order complete vector lattice and (x_n) a disjoint sequence in X. Then (x_n) uo-converges to 0 in X.*

Proof. Fix $x \in X_+$. We claim that $\sup_{k\geq n}(|x_k| \wedge x) \downarrow_n 0$. Indeed, suppose $\sup_{k\geq n}(|x_k| \wedge x) \geq y \geq 0$ for all $n \geq 1$. Then

$$0 \leq y \wedge |x_n| \leq \Big(\sup_{k\geq n+1}\big(|x_k| \wedge x\big)\Big) \wedge |x_n| = \sup_{k\geq n+1}\big(|x_k| \wedge |x_n| \wedge y\big) = 0.$$

Thus, $y \wedge |x_n| = 0$ for all $n \geq 1$. It follows that $y = y \wedge \sup_{n\geq 1}(|x_n| \wedge x) = \sup_{n\geq 1}(y \wedge |x_n| \wedge x) = 0$. This proves the claim. Now it is immediate that $|x_n| \wedge x \xrightarrow{o} 0$. $\square$

Recall that a vector $x > 0$ in a vector lattice X is called an atom if the ideal generated by x is one-dimensional, and that a vector lattice X is said to be atomic if the linear span of all atoms is order dense in X.

Lemma 1.2.

(1) *For a sequence (x_n) in a vector lattice X, if $x_n \xrightarrow{uo} 0$, then $\inf_k |x_{n_k}| = 0$ for any increasing sequence (n_k) of natural numbers.*
(2) *The converse holds true if X is an order complete atomic vector lattice or an order continuous Banach lattice.*

Proof. (1) Suppose $x_n \xrightarrow{uo} 0$. Fix any increasing sequence (n_k) of natural numbers. It is clear that $x_{n_k} \xrightarrow{uo} 0$. Let $x \in X$ be such that $|x_{n_k}| \geq x \geq 0$ for all $k \geq 1$. Then $x = |x_{n_k}| \wedge x \xrightarrow{o} 0$, implying that $x = 0$. Hence, $\inf_k |x_{n_k}| = 0$.

(2) Suppose $x_n \not\xrightarrow{uo} 0$ in X. Assume first that X is an order complete atomic vector lattice. Then X embeds as an ideal into $\mathbb{R}^A$ for some non-empty set A; cf. [9, Exercise 7, p. 143]. It follows from [4, Lemma 3.4] that $x_n \not\xrightarrow{uo} 0$ in $\mathbb{R}^A$. Therefore, $(x_n(a))$ does not converge to 0 for some point $a \in A$. Take an increasing sequence (n_k) such that $\inf_k |x_{n_k}(a)| > 0$. Then $\inf_k |x_{n_k}| > 0$ in $\mathbb{R}^A$, and thus, in X.

Assume now X is an order continuous Banach lattice. Let $x = \sum_1^\infty \frac{1}{2^n}\frac{|x_n|}{\|x_n\|+1}$. By [4, Lemma 3.4], $x_n \not\xrightarrow{uo} 0$ in the band B generated by x in X. Thus, by passing to B, we may assume that X has a weak unit. So there exists a strictly positive functional x^* on X; cf. [1, Theorem 4.15]. Let $\widetilde{X}$ be the completion of X with respect to the norm $\|x\|_L = x^*(|x|)$. We know that X is an ideal of the AL-space $\widetilde{X}$, which is lattice isometric to $L_1(\mu)$ for some measure μ; cf. [4, Subsection 2.2]. Hence, $x_n \not\xrightarrow{uo} 0$ in $\widetilde{X} = L_1(\mu)$. Equivalently, (x_n) does not converge to 0 almost everywhere. Thus, we can find an increasing sequence (n_k) such that $\inf_k |x_{n_k}| > 0$ in $\widetilde{X}$, and therefore, in X. $\square$