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Inclusions of Waterman–Shiba spaces into generalized Wiener classes

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ABSTRACT

The characterization of the inclusion of Waterman–Shiba spaces $\Lambda BV^{(p)}$ into generalized Wiener classes of functions $BV(q;\delta)$ is given. It uses a new and shorter proof and extends an earlier result of U. Goginava.

Let $\Lambda = (\lambda_i)$ be a Λ-sequence, that is, a nondecreasing sequence of positive numbers such that $\sum \frac{1}{\lambda_i} = +\infty$ and let p be a number greater than or equal to 1. A function $f : [0,1] \to \mathbb{R}$ is said to be of bounded p-Λ-variation if

$$V(f) := \sup \left(\sum_{i=1}^{n} \frac{|f(I_i)|^p}{\lambda_i} \right)^{\frac{1}{p}} < +\infty,$$

where the supremum is taken over all finite families $\{I_i\}_{i=1}^n$ of nonoverlapping subintervals of $[0,1]$ and where $f(I_i) := f(\sup I_i) - f(\inf I_i)$ is the change of the function f over the interval I_i. The symbol $\Lambda BV^{(p)}$ denotes the linear space of all functions of bounded p-Λ-variation with domain $[0,1]$. The Waterman–Shiba space $\Lambda BV^{(p)}$ was introduced in 1980 by M. Shiba [21]. When $p = 1$, $\Lambda BV^{(p)}$ is the well-known Waterman space ΛBV (see e.g. [25] and [26]). Some of the properties and applications of functions of class $\Lambda BV^{(p)}$ were discussed in [8,9,11,12,16–20,22–24]. $\Lambda BV^{(p)}$ equipped with the norm $\|f\|_{\Lambda,p} := |f(0)| + V(f)$ is a Banach space.

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H. Kita and K. Yoneda introduced a new function space which is a generalization of Wiener classes [14] (see also [13] and [1]). The concept was further extended by T. Akhobadze in [2] who studied many properties of the generalized Wiener classes $BV(q,\delta)$ thoroughly (see [3–7]).

Definition 1. Let $q = (q(n))_{n=1}^{\infty}$ be an increasing positive sequence and let $\delta = (\delta(n))_{n=1}^{\infty}$ be an increasing and unbounded positive sequence. We say that a function $f : [0,1] \to \mathbb{R}$ belongs to the class $BV(q;\delta)$ if

$$V(f,q;\delta) := \sup_{n\geq 1}\sup_{\{I_k\}}\left\{\left(\sum_{k=1}^{s}\left|f(I_k)\right|^{q(n)}\right)^{\frac{1}{q(n)}} : \inf_k |I_k| \geq \frac{1}{\delta(n)}\right\} < \infty,$$

where $\{I_k\}_{k=1}^{s}$ are non-overlapping subintervals of $[0,1]$.

If $\delta(n)^{1/q(n)}$ is a bounded sequence, then $BV(q;\delta)$ is simply the space of all bounded functions. This follows from the estimate

$$\left(\sum_{k=1}^{s}\left|f(I_k)\right|^{q(n)}\right)^{\frac{1}{q(n)}} \leq 2Cs^{\frac{1}{q(n)}} \leq 2C\delta(n)^{\frac{1}{q(n)}}, \quad \left|f(x)\right| \leq C.$$

The following statement regarding inclusions of Waterman spaces into generalized Wiener classes has been presented in [10]: if $\lim_{n\to\infty} q(n) = \infty$ and $\delta(n) = 2^n$ the inclusion $\Lambda BV \subset BV(q,\delta)$ holds if and only if

$$\limsup_{n\to\infty}\left\{\max_{1\leq k\leq 2^n}\frac{k^{\frac{1}{q(n)}}}{\left(\sum_{i=1}^{k}\frac{1}{\lambda_i}\right)}\right\} < +\infty. \tag{1}$$

Our result formulated below extends the above theorem of Goginava essentially and furnishes a new and much shorter proof.

Theorem 1. *For $p \in [1,\infty)$ and q and δ sequences satisfying the conditions in Definition 1, the inclusion $\Lambda BV^{(p)} \subset BV(q;\delta)$ holds if and only if*

$$\limsup_{n\to\infty}\left\{\max_{1\leq k\leq \delta(n)}\frac{k^{\frac{1}{q(n)}}}{\left(\sum_{i=1}^{k}\frac{1}{\lambda_i}\right)^{\frac{1}{p}}}\right\} < +\infty. \tag{2}$$

Before we present a relatively short proof of Theorem 1, we give an example showing that it provides a non-trivial extension of (1) even for $p = 1$. The example – provided by the referee kindly instead of our more complicated one – is obtained by taking $\lambda_n = n$, $q(n) = \sqrt{n}$ and $\delta(n) = 2^{\sqrt{n}}$. With those choices, it follows immediately that the Goginava indicator (1) is infinite while (2) holds.

Proof of Theorem 1. To show that (2) is a *sufficiency* condition for the inclusion $\Lambda BV^{(p)} \subset BV(q;\delta)$, we will prove the inequality

$$V(f,q,\delta) \leq V_{\Lambda BV^p}(f)\sup_n\left\{\max_{1\leq k\leq \delta(n)}\frac{k}{\left(\sum_{i=1}^{k} 1/\lambda_i\right)^{\frac{q(n)}{p}}}\right\}^{\frac{1}{q(n)}}. \tag{3}$$

This is a consequence of the numerical inequality

$$\sum_{j=1}^{s} x_j^q \leq \left(\sum_{j=1}^{s} x_j y_j\right)^q \max_{1\leq k\leq s}\frac{k}{\left(\sum_{j=1}^{k} y_j\right)^q}, \tag{4}$$