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Backward stochastic partial differential equations with quadratic growth

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A B S T R A C T

This paper is concerned with the weak solution (in analytic sense) to the Cauchy–Dirichlet problem of a backward stochastic partial differential equation when the nonhomogeneous term has a quadratic growth in both the gradient of the first unknown and the second unknown. Existence and uniqueness results are obtained under separate conditions.

1. Introduction

Let $\mathcal{D}$ be a simply connected and bounded domain in the Euclidean space $\mathbb{R}^d$. Denote by $\mathcal{T}$ the fixed time duration $[0,T]$. Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in\mathcal{T}}, \mathbb{P})$ be a complete filtered probability space on which a d_0-dimensional standard Wiener process $W = (W^1, \ldots, W^{d_0})^\top$ is defined such that $\{\mathscr{F}_t\}_{t\in\mathcal{T}}$ is the natural filtration generated by W and augmented by all the $\mathbb{P}$-null sets in $\mathscr{F}$. We denote by $\mathscr{P}$ the predictable σ-algebra associated with $\{\mathscr{F}_t\}_{t\in\mathcal{T}}$.

In this paper we consider the Cauchy–Dirichlet problem for the following super-parabolic quadratic backward stochastic partial differential equation (BSPDE in short)

$$du = -\left[\left(a^{ij}u_{x^j} + \sigma^{ik}q^k\right)_{x^i} + f(t,x,u,u_x,q)\right]dt + q^k dW_t^k, \tag{1.1}$$

with the terminal-boundary condition

$$\begin{cases} u(t,x) = 0, & t\in\mathcal{T},\ x\in\partial\mathcal{D}, \\ u(T,x) = \varphi(x), & x\in\mathcal{D}. \end{cases} \tag{1.2}$$

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Here we use the convention that repeated indices imply summation. By the terminology "super-parabolic" (cf. [22]) we mean the condition that there exist positive constants κ and K such that

$$\kappa I_d + (\sigma^{ik})(\sigma^{jk})^* \leq 2(a^{ij}) \leq K I_d. \tag{1.3}$$

And "quadratic" means that

$$\left|f(t,x,u,p,r)\right| \leq C\left(1 + |u| + |p|^2 + |r|^2\right), \tag{1.4}$$

for some positive constant C. We refer to (f, φ) as the *parameters* of BSPDE (1.1)–(1.2).

Historically speaking, BSPDEs, as a mathematically natural extension of backward SDEs (see e.g. [12, 24]) arise from many applications of probability theory and stochastic processes such as the optimal control of SDEs with incomplete information or stochastic parabolic PDEs (see e.g. [3,28,32]), and the formulation of the stochastic Feynman–Kac formula in mathematical finance (see e.g. [21]). A class of fully nonlinear BSPDEs, the so-called backward stochastic HJB equations, appears naturally in the dynamic programming theory of controlled non-Markovian processes (see e.g. [13,23,25]). For more aspects of BSPDEs, we refer to [1,30] and references therein.

A motivation of this work is from the theory of stochastic differential utility. Under uncertainty, Duffie and Epstein [11] introduced the following class of recursive utilities

$$V_t = \mathbb{E}\left[\int_t^T \left(f(c_t, V_t) + \frac{1}{2}A(V_s)\frac{d}{ds}[V]_s\right) ds \;\Big|\; \mathscr{F}_t\right], \quad t \in \mathcal{T},$$

where c is a consumption process and A is the "variance multiplier". Within a non-Markovian framework, an optimal consumption–investment problem based on such a kind of utilities is expected to give rise to a quadratic BSPDE (cf. [13,23]). Although there have been numbers of works devoted to the study of the wellposedness of BSPDEs (see e.g. [8–10,17,22,30,31]), only Lipschitz-type nonlinear BSPDEs have been studied in the literature as far as we know. This is the initiation of our current paper. Indeed, our result may not be directly applied to solve these consumption–investment problems, but it could be viewed as a first step towards the direction of analyzing quadratic BSPDEs.

The main idea of our approach is borrowed from the existing works on quadratic BSDEs. Such a kind of BSDEs arises from many fields, such as stochastic control problems with unbounded set of controls (see e.g. [4,29]), utility maximizations in incomplete markets (see e.g. [16]), and so on. In 2000, Kobylanski [18] obtained general existence and uniqueness results for one-dimensional quadratic BSDEs by utilizing the technique of Cole–Hopf transformation in PDE theory (cf. [2,5]). Her work has been extended and generalized by a series of works in the literature (see e.g. [6,7]). Although many ideas of [18] are used throughout this work, the difference in structure between quadratic BSDEs and BSPDEs requires additional arguments for the latter. Besides the fact that BSPDE is only solved in the sense of Schwartz-distribution, we should carefully deal with Itô's formula and the change of variables in this situation.

The rest of this paper is organized as follows. In Section 2 we introduce some notations and state main results. Several useful lemmas are collected in Section 3. Sections 4 and 5 are devoted to the proofs of uniqueness and existence theorems, respectively. In the final two sections, we present the proofs of two auxiliary propositions.

2. Notations and main results

Given a separable Banach space $\mathcal{B}$ and $p \in [1, \infty]$, we denote by $L^p_{\mathscr{P}}(\Omega \times \mathcal{T}; \mathcal{B})$ the space of all $\mathcal{B}$-valued predictable processes X such that