J. Math. Anal. Appl. 419 (2014) 496–524

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Mesh-independent a priori bounds for nonlinear elliptic finite difference boundary value problems

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ARTICLE INFO

Article history:
Received 4 April 2014
Available online 26 April 2014
Submitted by Goong Chen

Keywords:
Finite difference equations
Nonlinear boundary value problems
Critical exponent
A priori bounds

ABSTRACT

In this paper we prove mesh independent a priori L^∞-bounds for positive solutions of the finite difference boundary value problem

$$-\Delta_h u = f(x,u) \quad \text{in } \Omega_h, \qquad u = 0 \quad \text{on } \partial\Omega_h,$$

where Δ_h is the finite difference Laplacian and Ω_h is a discretized n-dimensional box. On the one hand this completes a result of [9] on the asymptotic symmetry of solutions of finite difference boundary value problems. On the other hand it is a finite difference version of a critical exponent problem studied in [10]. Two main results are given: one for dimension $n = 1$ and one for the higher dimensional case $n \geq 2$. The methods of proof differ substantially in these two cases. In the 1-dimensional case our method resembles ode-techniques. In the higher dimensional case the growth rate of the nonlinearity has to be bounded by an exponent $p < \frac{n}{n-1}$ where we believe that $\frac{n}{n-1}$ plays the role of a critical exponent. Our method in this case is based on the use of the discrete Hardy–Sobolev inequality as in [3] and on Moser's iteration method. We point out that our a priori bounds are (in principal) explicit.

1. Introduction

The purpose of this paper is two-fold. The first aim is motivated by the general principle that when there exist major results for a semilinear elliptic boundary value problem, then if we formulate a reasonable discretization of this boundary value problem with a view to finding approximate solutions, then there

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http://dx.doi.org/10.1016/j.jmaa.2014.04.044
0022-247X/

should exist analogous results for the corresponding discretized problem. A typical example of this idea may be found in [9].

A well-known theorem of Gidas, Ni, and Nirenberg, [4], states roughly that positive solutions of the semi linear elliptic boundary value problem

$$-\Delta u = f(u), \quad x \in \Omega, \qquad u(x) = 0, \quad x \in \partial\Omega \tag{1}$$

inherit symmetries of the domain Ω. For example, if Ω is a ball, then all positive solutions must be radially symmetric. If Ω is a hypercube, then all positive solutions must be symmetric about the bisecting hyperplanes.

It is natural to ask whether there is a corresponding result for the corresponding discretized problem. In other words, if we replace the Laplacian in (1) with the corresponding finite difference Laplacian ($e_1, \dots, e_n$ is the standard basis of $\mathbb{R}^n$ and $h_1, \dots, h_n > 0$ stand for the mesh sizes)

$$\Delta_h u(x) := \sum_{i=1}^{n} \frac{u(x + h_i e_i) - 2u(x) + u(x - h_i e_i)}{h_i^2}$$

we obtain the following finite difference version of (1)

$$-\Delta_h u = f(u), \quad x \in \Omega_h, \qquad u(x) = 0, \quad x \in \partial\Omega_h. \tag{2}$$

The question is: does a discrete solution satisfy the same type of symmetries? (Assuming, of course, that the discretized grid reflects these symmetries.) The answer is no, even in one dimension. Easy counterexamples of this can be found in [9]. The problem is that there is no restriction on the mesh size. One should only expect the (positive) solutions of Eq. (2) to reflect those of Eq. (1) when the mesh sizes are *small*.

This gives a clue to the correct result, also in [9], which can be roughly summarized as follows; *as the space step of the discretization becomes small, the solutions u_h become approximately symmetric*. Concrete estimates of the distance from symmetry are given in terms of the difference between the solution and its reflection about the bisecting hyperplane. Full details can be found in [9]. This is an example of the general principle mentioned above. If the discretization is sufficiently fine, the properties of the continuous solutions of Eq. (1) should be reflected in the properties of the discretized solutions of Eq. (2).

Of course, several technical assumptions, both on the Lipschitz constants for f and the behavior of the approximate solutions u_h are required. One key assumption was that there exists $M > 0$ such that $\|u_h\|_\infty \leq M$. Since this can often be obtained for a wide variety of nonlinearities via the discrete maximum principle, we felt, at the time, that this assumption was not unreasonable, and indeed natural in the numerical context. After all, if $\|u_h\|_\infty \to \infty$ as a subsequence of the $h \to 0$, one would naturally think that we were not in the neighborhood of a true classical or weak solution. However, as recently observed in [10] and [7], the blow-up of the $\|\cdot\|_\infty$-norm of a family of finite-difference solutions may indicate the existence of an unbounded distributional solution induced by a supercritical exponent in the nonlinearity, cf. Remark (a) after Theorem 2.

However, there is another large class of nonlinearities, to which the maximum principle is not applicable, but for which the true solutions of (1) can be shown to satisfy a priori bounds. A summary of these results can be found in [10]. To extend some of the results of [9] to a discrete setting is the second aim of this paper.

These are the two goals mentioned at the beginning of this introduction. First, proving these a priori estimates will achieve the aim of proving corresponding results for solutions of the discretized problems. Second, it will extend to a much wider class of nonlinearities the approximate symmetry results of [9].

A priori estimates will be proven for positive solutions of the following generalization of (2)

$$-\Delta_h u = f(x, u) \quad \text{in } \Omega_h, \qquad u = 0 \quad \text{on } \partial\Omega_h \tag{3}$$