J. Math. Anal. Appl. 419 (2014) 541–552

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Some new convergent sequences and inequalities of Euler's constant

Dawei Lu

School of Mathematical Sciences, Dalian University of Technology, Dalian 116023, China

ARTICLE INFO

Article history:
Received 9 January 2014
Available online 14 May 2014
Submitted by B.C. Berndt

Keywords:
Euler's constant
Rate of convergence
Asymptotic expansion

ABSTRACT

In this paper, some new convergent sequences and inequalities of Euler's constant are provided. To demonstrate the superiority of our new convergent sequence over DeTemple's sequence, Vernescu's sequence and Mortici's sequences, some numerical computations are also given.

1. Introduction

In the theory of mathematical constants, an important concern is the definition of new sequences which converge to these fundamental constants with increasingly higher speed. These convergent sequences and constants play a key role in many areas of mathematics and science in general, as theory of probability, applied statistics, physics, special functions, number theory, or analysis.

One of the most useful convergent sequences in mathematics is

$$\gamma_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n, \tag{1.1}$$

which converges towards the well-known Euler's constant

$$\gamma = 0.57721566490115328\ldots.$$

Up until now, many researchers made great efforts in the area of concerning the rate of convergence of the sequence $(\gamma_n)_{n\geqslant 1}$ and establishing faster sequences to converge to Euler's constant and had a lot of inspiring results. For example, in [11–13,15], the following estimates are established

$$\frac{1}{2n+1} < \gamma_n - \gamma < \frac{1}{2n}, \tag{1.2}$$

E-mail address: ludawei_dlut@163.com.

http://dx.doi.org/10.1016/j.jmaa.2014.05.018
0022-247X/

using interesting geometric interpretations. In [14], Vernescu provided the sequence

$$V_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{2n} - \ln n, \tag{1.3}$$

for which

$$\frac{1}{12(n+1)^2} < \gamma - V_n < \frac{1}{12n^2}. \tag{1.4}$$

In [1,2], DeTemple introduced a faster convergent sequence $(R_n)_{n\geqslant 1}$ to γ as follows,

$$R_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln\left(n + \frac{1}{2}\right), \tag{1.5}$$

which decreases to γ with the rate of convergence n^{-2}, since

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}. \tag{1.6}$$

Both (1.3) and (1.5) are slight modifications of Euler's sequences (1.1), but significantly improve the rate of convergence from n^{-1} to n^{-2}.

Recently, Mortici researched Euler's constant again, and provided some convergent sequences which are faster than (1.1), (1.3) and (1.5).

In [4], Mortici provided the following two sequences

$$u_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{(6-2\sqrt{6})n} - \ln\left(n + \frac{1}{\sqrt{6}}\right) \tag{1.7}$$

and

$$v_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{(6+2\sqrt{6})n} - \ln\left(n - \frac{1}{\sqrt{6}}\right). \tag{1.8}$$

Both sequences (1.7) and (1.8) were shown to converge to γ as n^{-3}.

Next, in [6], Mortici introduced the following class of sequences of the form

$$\mu_n(a,b) = \sum_{k=1}^{n} \frac{1}{k} + \ln\left(e^{a/(n+b)} - 1\right) - \ln a, \tag{1.9}$$

where a, b are real parameters, $a > 0$. Furthermore, they proved that among the sequences $(\mu_n(a,b))_{n\geqslant 1}$, the privileged one

$$\mu_n\left(\frac{\sqrt{2}}{2}, \frac{2+\sqrt{2}}{4}\right)$$

offers the best approximations of γ, since

$$\lim_{n\to\infty} n^3\left(\mu_n\left(\frac{\sqrt{2}}{2}, \frac{2+\sqrt{2}}{4}\right) - \gamma\right) = \frac{\sqrt{2}}{96}. \tag{1.10}$$