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Dynamics of stochastic three dimensional Navier–Stokes–Voigt equations on unbounded domains

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A B S T R A C T

The aim of this paper is to study the long time behavior of the following stochastic 3D Navier–Stokes–Voigt equation

$$u_t - \nu\Delta u - \alpha^2\Delta u_t + (u\cdot\nabla)u + \nabla p = g(x) + \varepsilon h\frac{d\omega}{dt}$$

in an arbitrary (bounded or unbounded) domain satisfying the Poincaré inequality. By famous J. Ball's energy equation method, we obtain a unique random attractor $\mathcal{A}_\varepsilon$ for the random dynamical system generated by the equation. Moreover, we prove that the random attractor $\mathcal{A}_\varepsilon$ tends to the global attractor $\mathcal{A}_0$ of the deterministic equation in the sense of Hausdorff semi-distance as $\varepsilon \to 0$.

1. Introduction

Let $\mathcal{O}$ be an arbitrary domain (bounded or unbounded) in $\mathbb{R}^3$ with smooth boundary $\partial\mathcal{O}$, in which the Poincaré inequality holds

$$\int_{\mathcal{O}} |\nabla\varphi|^2 dx \ge \lambda_1 \int_{\mathcal{O}} |\varphi|^2 dx. \tag{1.1}$$

We consider the following stochastic 3D Navier–Stokes–Voigt (NSV for short) equation

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$$\begin{cases} u_t - \nu\Delta u - \alpha^2\Delta u_t + (u\cdot\nabla)u + \nabla p = g(x) + \varepsilon h\dfrac{d\omega}{dt}, \quad x \in \mathcal{O},\ t > 0, \\ \nabla\cdot u = 0, \quad x \in \mathcal{O},\ t > 0, \\ u(x,t) = 0, \quad x \in \partial\mathcal{O},\ t > 0, \\ u(x,0) = u_0(x), \quad x \in \mathcal{O}, \end{cases} \tag{1.2}$$

where $u = (u_1, u_2, u_3)$ is the unknown velocity vector, $p = p(x,t)$ is the unknown pressure, $\nu > 0$ and α are viscosity coefficient and length scale parameter characterizing the elasticity of the fluid, respectively; $h \in (H_0^1(\mathcal{O}))^3$ and ω is a two-sided real-valued Wiener process on a probability space which will be specified later; $\varepsilon \in (0,1]$ is a small parameter which describes how much of random noise that presents in the equation.

The system (1.2) is presented in [8] as a regularization of the 3D Navier–Stokes equations for the sake of direct numerical simulations for either the period or the no-slip Dirichlet boundary conditions. The system also models the dynamics of a Kelvin–Voigt viscoelastic incompressible fluid and was introduced in [18] as a model of motion of linear, viscoelastic fluids. When $\alpha = 0$, it turns out to be the classical 3D Navier–Stokes system.

In deterministic case, that is $\varepsilon = 0$, the long time behavior of (1.2) has been widely studied. For example, in autonomous case of deterministic equations, Kalantarov [15,16] proved the existence of a finite dimensional global attractor for the corresponding semigroup. The Gevrey regularity for the global attractor was shown in [17] by Kalantarov and Titi. In non-autonomous case, the uniform attractor and pullback attractor were recently obtained in [25] and [1], respectively.

The large time asymptotic of the stochastic case, i.e. $\varepsilon > 0$, however, is not well understood. Up to the best of our knowledge, there are only two results in this direction: the existence of a random attractor was proved recently in [13], and the Hausdorff dimension for this attractor was given in [20]. It is worth noting that both of the above papers considered NSV equation in bounded domains. In this work, for the first time, we study the long time behavior of stochastic 3D NSV equation (1.2) in unbounded domains.

The first aim of the present paper is to prove the existence of a unique random attractor $\mathcal{A}_\varepsilon$ for the stochastic NSV equation (1.2) for each $\varepsilon \in (0,1]$. The concept of random attractor, which captures the long time behavior of random dynamical systems generated by stochastic equations, was introduced in [12] and has been studied extensively by many authors, see e.g. [4,5,13,24,22] and references therein. The existence of random attractors usually heavily depends on the existence of a compact random absorbing set, see e.g. [12]. This is often done for an equation which has smoothing effect, that says the solution of the equation is more regular than initial data, such as reaction–diffusion equation or Navier–Stokes equation. Unfortunately, in our case, due to the presence of the term $-\alpha^2\Delta u_t$, if the initial datum $u_0 \in (H_0^1(\mathcal{O}))^3$ then the solution always belongs to $(H_0^1(\mathcal{O}))^3$ and has no higher regularity. This issue can be solved by a splitting solution method (see e.g. [13,17]), that is the solution is divided into two parts in which the first part decays to zero while the second part belongs to higher regular space which compactly embeds to space of initial data. However, this method doesn't work in the present paper since we do not have such a compact embedding in unbounded domains. To overcome this difficulty, we will use the so-called energy equation method, which was introduced by Ball [3] and successfully applied to deterministic equations [1,14,19] as well as stochastic equations [7,6,24]. Compare to [7,6], in which authors proved the existence of random attractors for 2D Navier–Stokes equations in unbounded domains, the new point of this work is that we deal with three dimensional case (then of course it is valid in two dimensional case) and different context of equation because of the term $-\alpha^2\Delta u_t$. Nevertheless, the noise we consider here is more regular than that of [7,6]. The long time asymptotic of (1.2) with rough noises is more involved and can be treated in a different work.

The second aim of this work is to show the upper semi-continuity of the family of random attractors $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in(0,1]}$ at $\varepsilon = 0$. That is, when $\varepsilon \to 0$, the random attractor $\mathcal{A}_\epsilon$ corresponding to (1.2) tends to the global attractor $\mathcal{A}_0$ of deterministic NSV equation. Upper semi-continuity of random attractors for dynamical