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Some isoperimetric inequalities and eigenvalue estimates in weighted manifolds

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ABSTRACT

In this paper we prove general inequalities involving the weighted mean curvature of compact submanifolds immersed in weighted manifolds. As a consequence we obtain a relative linear isoperimetric inequality for such submanifolds. We also prove an extrinsic upper bound to the first non-zero eigenvalue of the drift Laplacian on closed submanifolds of weighted manifolds.

1. Introduction

Let $(\bar{M}^d, \bar{g}, d\bar{\mu})$ be a weighted manifold, that is, a Riemannian manifold $(\bar{M}^d, \bar{g})$ endowed with a weighted volume form $d\bar{\mu} = e^{-f}\, d\bar{M}$, where f is a real-valued smooth function on $\bar{M}$ and $d\bar{M}$ is the volume element induced by the metric $\bar{g}$.

In weighted manifolds a natural generalization of the Ricci tensor is the m-Bakry–Émery tensor defined by

$$\overline{Ric}_f^m = \overline{Ric} + \bar{\nabla}^2 f - \frac{1}{m-d}\, df \otimes df,$$

for each $m \in [d, \infty)$. When $m = \infty$ it gives the tensor $Ric_f = Ric + \bar{\nabla}^2 f$ introduced by Lichnerowicz [9,10] and independently by Bakry and Émery in [1]. The case $m = d$ only makes sense when the function f is constant and so $\overline{Ric}_f^m$ is the usual Ricci tensor $\overline{Ric}$ of $\bar{M}$.

In this paper we are interested in studying inequalities on submanifolds of weighted manifolds. In order to do it we make use of intrinsic objects, like the m-Bakry–Émery tensor, and extrinsic objets like the weighted

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mean curvature defined below. Namely, given $x : M \to \bar{M}$ an isometric immersion, we define the weighted mean curvature vector $\mathbf{H}_f$ by

$$\mathbf{H}_f = \mathbf{H} + \bar{\nabla} f^{\perp},$$

where $\mathbf{H}$ is the mean curvature vector of the submanifold M and $^{\perp}$ denotes the orthogonal projection onto the normal bundle $TM^{\perp}$ (see Gromov [5] and Morgan [12]). The weighted mean curvature appears naturally in the first variation of the weighted area functional as described in [2]. In the submanifold M we also consider the weighted volume given $d\mu = e^{-f}\, dM$, where dM is the volume element of M.

In case that $\bar{M} = \Omega^{n+1}$, where Ω is a compact oriented $(n+1)$-dimensional Riemannian manifold with smooth boundary $M^n = \partial\Omega$ we consider on M the Riemannian metric induced by the inclusion map $\iota : M \hookrightarrow \Omega$.

Let ν be a unit normal vector field on M and let A denote the shape operator of M, that is $A = -\nabla_{(.)}\nu$. It is easy to see that $\mathbf{H}_f = H_f \nu$, where $H_f = H + \langle \bar{\nabla} f, \nu \rangle$ and $H = \operatorname{trace} A$ is the mean curvature function.

In [15], Ros proved an inequality relating the volume of Ω and the mean curvature function H of M. The inequality obtained by Ros is essentially contained in the paper of Heintze and Karcher [7], although the proof uses different techniques.

Our first result is the natural generalization of Ros inequality in the context of weighted manifolds.

Theorem 1.1. *Let Ω^{n+1} be a compact weighted manifold with smooth boundary M and non-negative m-Bakry–Émery tensor. Let H_f be the weighted mean curvature of M. If H_f is positive everywhere, then*

$$Vol_f(\Omega) \leqslant \frac{m-1}{m} \int\limits_M \frac{1}{H_f}\, d\mu.$$

Moreover, equality holds if and only if Ω is isometric to a Euclidean ball, f is constant and $m = n+1$.

Extending the Ros formula, Choe and Park [4] proved that a compact connected embedded CMC hypersurface in a convex Euclidean solid cone which is perpendicular to the boundary of the cone is part of a round sphere.

The rigidity of compact submanifold with free boundary is a very classical problem in submanifold theory. For instance, Nitsche [13] proved that an immersed disk type constant mean curvature surface in a ball which makes a constant angle with the boundary of the ball is part of a round sphere.

On weighted manifolds, Cañete and Rosales [3] showed the rigidity of compact *stable* hypersurfaces with free boundary in a convex solid cone in Euclidean space with homogeneous density. Our next result extends Choe and Park's result to weighted Euclidean spaces $(\mathbb{R}^{n+1}, ds_0, d\bar{\mu})$, where ds_0 is the Euclidean metric.

Theorem 1.2. *Let C be a convex solid cone with piecewise smooth boundary ∂C in a weighted manifold $(\mathbb{R}^{n+1}, ds_0, d\bar{\mu})$ of non-negative m-Bakry–Émery tensor. Let M be a compact connected embedded hypersurface in C and Ω the bounded domain enclosed by M and ∂C. If the weighted mean curvature H_f of M is positive everywhere, then*

$$Vol_f(\Omega) \leqslant \frac{m-1}{m} \int\limits_M \frac{1}{H_f}\, d\mu.$$

Moreover, the equality holds if and only if M is part of a round sphere centered at the vertex of C and f is constant and $m = n+1$.