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Uniqueness result for non-negative solutions of semi-linear inequalities on Riemannian manifolds

Yuhua Sun [1]

Department of Mathematics, University Bielefeld, 33501 Bielefeld, Germany

A R T I C L E I N F O

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We consider certain semi-linear partial differential inequalities on complete connected Riemannian manifolds and provide a simple condition in terms of volume growth for the uniqueness of a non-negative solution. We also show the sharpness of this condition.

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1. Introduction

Consider a geodesically complete connected manifold M and the following differential inequality on M

$$div\big(A(x)\nabla u\big) + V(x)u^\sigma \le 0, \tag{1.1}$$

where div and ∇ are the Riemannian divergence and gradient respectively, $A(x)$ is a non-negative definite symmetric operator in the tangent space T_xM, such that $x \mapsto A(x)$ is measurable, V is a given locally integrable positive measurable function, and $\sigma > 1$ is a given constant.

Our purpose is to provide simple geometric condition on M to ensure that the only non-negative solution u of (1.1) is identical zero.

This problem has a long history. For the inequality

$$\Delta u + u^\sigma \le 0, \tag{1.2}$$

E-mail address: ysun@math.uni-bielefeld.de.

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in $\mathbb{R}^n$, the following result was proved by Ni and Serrin, Caristi and Mitidieri (cf. [2,18,19]): In the case $n \geq 3$, the only non-negative solution of (1.2) is identical zero if and only if $\sigma \leq \frac{n}{n-2}$, while in the case $n \leq 2$, the same is true for any σ.

Note for exact equality $\Delta u + u^\sigma = 0$, the critical value of parameter σ is different and is equal to $\frac{n+2}{n-2}$, when $n > 2$ (cf. [8]).

A more general inequality of (1.1) in $\mathbb{R}^n$ and even more complicated inequalities and equations have been thoroughly studied in a series of papers by Mitidieri, Pohozaev [12–17], D'Ambrosio, Mitidieri, Pohozaev [5–7], and Caristi, D'Ambrosio, Mitidieri [1–3]. They have developed a universal method of proving uniqueness for non-negative solutions, which is based on capacity estimates, which in turn rely on suitable choice of test functions.

In this paper we also use the method of test functions. In fact our proof up to (2.15) follows the same argument as in [15] and other papers cited in the above paragraph. However, after that we make a different choice of test function that enables us to work with minimal geometric assumption about the underlying manifold, namely, with the restriction on the volume growth of geodesic balls.

One of the difficulties that arises in the setting of manifold is that it is not possible to produce test function φ with suitable estimate of $L\varphi$, where $L = div(A\nabla\cdot)$. More precisely, estimate of this kind would require restrictions of the curvature of the manifold, which we avoid. In fact, the only geometric assumption that we impose on manifold is the volume growth restriction. Note that under this mild assumption the commonly used in PDEs estimates of the fundamental solution, Harnack inequalities etc. are not available.

Denote by μ the Riemannian measure on M and by $B(x, r)$ the geodesic ball on M of radius r centered at $x \in M$. Given that $d(\cdot,\cdot)$ is geodesic distance, and μ is the Riemannian measure. Assume that $V(x) \in L^1_{loc}(M, \mu)$ throughout the paper.

Cheng and Yau proved that if for some $x_0 \in M$ and all large enough r

$$\mu\big(B(x_0, r)\big) \leq Cr^2, \tag{1.3}$$

then any positive solution to $\Delta u \leq 0$ is identical constant (cf. [4]).

Grigor'yan and the author proved in [10] that when

$$\mu\big(B(x_0, r)\big) \leq Cr^p \ln^q r, \tag{1.4}$$

where

$$p = \frac{2\sigma}{\sigma - 1}, \qquad q = \frac{1}{\sigma - 1}. \tag{1.5}$$

Then the only non-negative solution of (1.2) is identical zero. Note that (1.2) is a particular case of (1.1) for $A(x) = Id$ and $V(x) = 1$. Moreover, they constructed an example to show the sharpness of the exponents p and q, that is, if either $p > \frac{2\sigma}{\sigma-1}$, or $p = \frac{2\sigma}{\sigma-1}$ and $q > \frac{1}{\sigma-1}$, then there is a manifold satisfying the volume estimate (1.4), and (1.2) has a non-trivial non-negative solution.

For general A, V in (1.1), Grigor'yan and Kondratiev in [9] used measure ν_ϵ defined for any $\epsilon > 0$ by

$$d\nu_\epsilon = \|A\|^{\frac{\sigma}{\sigma-1}-\epsilon} V^{-\frac{1}{\sigma-1}+\epsilon} d\mu, \quad \text{for } \epsilon > 0,$$

and proved that if

$$\nu_\epsilon\big(B(x_0, r)\big) \leq Cr^{p+C\epsilon} \ln^\kappa r, \tag{1.6}$$

holds for some $\kappa < q$ and all small enough $\epsilon > 0$, where p, q are given by (1.5). Then any non-negative solution of (1.1) is identically equal to zero. Some conditions for uniqueness of non-negative solutions in terms of capacities were proved in [9,11].

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