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Nontrivial periodic motions for resonant type asymptotically linear lattice dynamical systems ☆

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ABSTRACT

In this paper, we consider the following one dimensional lattices consisting of infinitely many particles with nearest neighbor interaction

$$\ddot{q}_i(t) = \Phi'_{i-1}\big(t, q_{i-1}(t) - q_i(t)\big) - \Phi'_i\big(t, q_i(t) - q_{i+1}(t)\big), \quad i \in \mathbb{Z},$$

where $\Phi_i(t,x) = -(\alpha_i/2)|x|^2 + V_i(t,x)$ is T-periodic in t for $T > 0$ and satisfies $\Phi_{i+N} = \Phi_i$ for some $N \in \mathbb{N}$, $q_i(t)$ is the state of the i-th particle. Assume that $\alpha_i = 0$ for some $i \in \mathbb{Z}$ and $V'_i(t,x)$ denoting the derivative of V_i respect to x is asymptotically linear with x both at origin and at infinity. We would like to point out that this system is resonant both at origin and at infinity and not studied up to now. Based on some new results concerning the precise computations of the critical groups, for a given $m \in \mathbb{Z}$, we obtain the existence of nontrivial periodic solutions satisfying $q_{i+mN}(t+T) = q_i(t)$ for all $t \in \mathbb{R}$ and $i \in \mathbb{Z}$ under some additional conditions.

1. Introduction and main results

In this paper, we consider one dimensional lattices consisting of infinitely many particles with nearest neighbor interaction. We represent the state of the non-autonomous dynamical system at time t by a sequence of functions $q(t) = \{q_i(t)\}$, $i \in \mathbb{Z}$, where $q_i(t)$ is the state of the i-th particle. Let $\Phi_i(t,\cdot)$ denote the potential of the interaction between the i-th and the $(i+1)$-th particle (whose displacement is $q_i(t) - q_{i+1}(t)$), then the equation governing the state of $q_i(t)$ reads

$$\ddot{q}_i(t) = \Phi'_{i-1}\big(t, q_{i-1}(t) - q_i(t)\big) - \Phi'_i\big(t, q_i(t) - q_{i+1}(t)\big), \quad i \in \mathbb{Z}, \tag{1.1}$$

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where $t \in \mathbb{R}$. Here and in the sequel $\Phi_i'(t,x)$ denotes the derivative of Φ_i respect to x. We define the potential $\Phi : \mathbb{R} \times \mathbb{R}^\infty \to \mathbb{R}$ by

$$\Phi\big(t, q(t)\big) = \sum_{i=-\infty}^{+\infty} \Phi_i\big(t, q_i(t) - q_{i+1}(t)\big), \quad t \in \mathbb{R}.$$

Then infinitely many equations (1.1) can be written in a vectorial form

$$\ddot{q}(t) = -\Phi'\big(t, q(t)\big), \quad t \in \mathbb{R}. \tag{1.2}$$

We first recall some historical comments on related work. After the pioneering numerical experiment of Fermi, Pasta and Ulam [9] on finite lattices, an autonomous dynamical system with finitely or infinitely many degrees of freedom whose dynamics is described by the equations

$$\ddot{q}_i = \Phi_{i-1}'(q_{i-1} - q_i) - \Phi_i'(q_i - q_{i+1}), \quad i \in \mathbb{Z}, \tag{1.3}$$

has been widely studied under different kinds of potentials [19]. Let us now briefly recall the main results obtained before. Due to the implementation of variational methods, a number of rigorous results was obtained in the case of the general equations (1.3) in the 1990s. In [10,16,18], travelling waves, i.e. solutions of the form $q_i(t) = u(i - ct)$ were studied by using a constrained minimization approach, Nehari manifold approach and mountain pass theorem, respectively. We would like to point out the paper [17] considering the existence of infinitely many travelling waves of multibump type for the non-autonomous case.

The existence and multiplicity of periodic motions for system (1.3) have been studied, restricting the system to periodic potentials, that is $\Phi_i = \Phi_{i+N}$ for some integer N. In [2] a nontrivial solution is obtained as a mountain pass point for the corresponding Lagrangian functional, under the assumption that $\Phi_i(x) := -\alpha_i x^2 + V_i(x)$, satisfying $\alpha_i > 0$ and $V_i(x) \geqslant 0$ for all $i \in \mathbb{Z}$, is quadratically repulsive for small displacements and superquadratically attractive for large displacements; note that $\alpha_i > 0$ implies the minimum of the spectrum of the quadratic part of the Lagrangian functional is strictly positive. In [1], Arioli and Gazzola extended the result to the purely attractive potentials ($\alpha_i = 0$ for all $i \in \mathbb{Z}$) which are strictly superquadratic at both the origin and the infinity; in this case the minimum of the spectrum is 0. In [4] the existence of infinitely many periodic non-constant solutions of multibump type has been proved, with the same assumptions taken in [2]. In [3], a nonzero periodic solutions of finite energy has been obtain with to the potentials quadratically attractive and the coefficients α_i take both signs.

We assume that the potentials Φ_i are given by

$$\Phi_i(t,x) = -\frac{\alpha_i}{2}|x|^2 + V_i(t,x),$$

where $V_i(t,x)$ is C^2 in x and T-periodic in t for some $T > 0$. Similar to [3] (also [1]), if $\alpha_i = 0$ for some i, it is easy to check that 0 lies in the spectrum of the quadratic part of the Lagrangian functional corresponding to system (1.2) (see Remark 2.1 below for more details). Thus a nature question is whether system (1.2) has nonzero periodic solutions when $V_i'(t,x)$ is asymptotically linear respect to x both at origin and at infinity; in other words, this problem is resonant both at origin and at infinity. To the best of our knowledge, resonant type asymptotically linear lattice dynamical systems have not been studied up to now. Furthermore $\alpha_i = 0$ for some i is "necessary" if one wants to study resonant type problem (1.2) in some sense, since the properties of the spectrum except 0 are not very clear. In this paper, we give a positive answer to this question. It is worth mentioning that the autonomous case can be treated similarly.

Before we state our main results, we give some assumptions on the potentials Φ_i.

(Φ) There exists $N \in \mathbb{N}$ such that $\Phi_{i+N} = \Phi_i$;

(Φ^0) There exists constant $\beta \in [1, +\infty)$ such that $\lim_{|x|\to 0} \frac{|V_i'(t,x)|}{|x|^\beta} = 0$ uniformly for $t \in [0,T]$;