J. Math. Anal. Appl. 417 (2014) 635–642

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Continuity properties of the data-to-solution map for the generalized Camassa–Holm equation

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ARTICLE INFO

Article history:
Received 19 December 2013
Available online 25 March 2014
Submitted by C. Gutierrez

Keywords:
Camassa–Holm equation
Degasperis–Procesi equation
Novikov equations
Integrable equations
Peakon solutions
Cauchy problem
Fourier transform
Sobolev spaces
Hölder continuity
Commutator estimate
Multiplier estimates

ABSTRACT

This work studies a generalized Camassa–Holm equation with higher order non-linearities (g-kbCH). The Camassa–Holm, the Degasperis–Procesi and the Novikov equations are integrable members of this family of equations. g-kbCH is well-posed in Sobolev spaces H^s, $s > 3/2$, on both the line and the circle and its solution map is continuous but not uniformly continuous. In this work it is shown that the solution map is Hölder continuous in H^s equipped with the H^r-topology for $0 \leqslant r < s$, and the Hölder exponent is expressed in terms of s and r.

1. Introduction and results

We consider the Cauchy problem for the generalized Camassa–Holm (g-kbCH) equation

$$\begin{cases} (1-\partial_x^2)\partial_t u = u^k\partial_x^3 u + bu^{k-1}\partial_x u\partial_x^2 u - (b+1)u^k\partial_x u, \\ u(x,0) = u(0), \quad x \in \mathbb{R} \text{ or } \mathbb{T} \text{ and } t \in \mathbb{R}, \end{cases} \tag{1.1}$$

and prove that for initial data u_0 in the Sobolev space H^s, $s > 3/2$, the data-to-solution map $u(0) \mapsto u(t)$ is Hölder continuous in H^r-topology for all $0 \leqslant r < s$. Well-posedness of g-kbCH on both the line and the circle was proved in Himonas and Holliman [14]. More precisely, the authors show that for a given initial data $u_0 \in H^s$ there exists a lifespan

$$0 < T < \frac{1}{2kc_s\|u_0\|_{H^s}^k},$$

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http://dx.doi.org/10.1016/j.jmaa.2014.03.033
0022-247X/

where $c_s > 0$ is a constant depending upon s, and a unique solution $u \in C([0,T];H^s) \cap C^1([0,T];H^{s-1})$, of the Cauchy problem (1.1) such that

$$\|u(t)\|_{H^s} \leqslant 2\|u_0\|_{H^s}. \tag{1.2}$$

Furthermore, they show that the data-to-solution map is continuous but not uniformly continuous on any bounded subset of H^s.

This result employed a Galerkin type approximation (see Taylor [31]). The same methodology was also employed by [12] for the quadratic Degasperis–Procesi (DP) equation (corresponding to $k = 1$ and $b = 3$)

$$(1 - \partial_x^2)\partial_t u = u\partial_x^3 u + 3\partial_x u\partial_x^2 u - 4u\partial_x u, \tag{1.3}$$

which was discovered in 1999 [8], and the cubic Novikov (NE) equation [13] ($k = 2$, $b = 3$), discovered in 2009 [29]

$$(1 - \partial_x^2)\partial_t u = u^2\partial_x^3 u + 3u\partial_x u\partial_x^2 u - 4u^2\partial_x u. \tag{1.4}$$

The most famous member of the g-kbCH family, however, is the Camassa–Holm (CH) equation ($k = 1$, $b = 2$)

$$(1 - \partial_x^2)\partial_t u = u\partial_x^3 u + 2\partial_x u\partial_x^2 u - 3u\partial_x u. \tag{1.5}$$

These three equations are all integrable and have been extensively studied in the literature. They can be written in the form

$$(1 - \partial_x^2)\partial_t u = P(u, \partial_x u, \partial_x^2 u, \partial_x^3), \tag{1.6}$$

where P is a polynomial (see Novikov [29] for a complete list of integrable equations of this form when P is a cubic or quadratic polynomial). Also, they possess infinitely many conserved quantities, an infinite hierarchy of quasi-local symmetries, a Lax pair, and a bi-Hamiltonian structure. The g-kbCH equation is a natural unifier for CH, DP and NE.

Written in its nonlocal form (see (2.9) below), the g-kbCH equation can be thought as a weakly dispersive perturbation of the generalized Burgers equation $\partial_t u + u^k\partial_x u = 0$. However, while this equation has no peakon solutions the g-kbCH equation does, for all values of k and b. This is another property making g-kbCH an interesting evolution equation. In the non-periodic case the g-kbCH peakon solutions are

$$u_c(x,t) = c^{1/k}e^{-|x-ct|}, \tag{1.7}$$

where c is a positive constant (see Grayshan and Himonas [11]). This result followed earlier works deriving peakon solutions for CH, DP and NE. For CH, the first peakon solution was written down by Camassa and Holm in [3]. Lenells [26] provided a classification of all traveling wave solutions for CH. Degasperis Holm and Hone [9] provided the peakon for DP (see also [22] and [27]). Furthermore, Hone and Wang [23] and Hone, Lundmark and Szmigielski [24] produced multipeakon solutions for NE. We mention that the discovery of CH in 1993 by Camassa and Holm [3] was in part driven by the desire to find a water wave equation which allows for wave breaking. In this work, the CH equation is derived as an approximation to the Euler equations of ideal fluids. In contrast, the KdV equation

$$\partial_t u + 6u\partial_x u + \partial_x^3 u = 0, \tag{1.8}$$