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Folding and unfolding in periodic difference equations

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ABSTRACT

Given a p-periodic difference equation $x_{n+1} = f_{n \bmod p}(x_n)$, where each f_j is a continuous interval map, $j = 0, 1, \ldots, p-1$, we discuss the notion of folding and unfolding related to this type of non-autonomous equations. It is possible to glue certain maps of this equation to shorten its period, which we call folding. On the other hand, we can unfold the glued maps so the original structure can be recovered or understood. Here, we focus on the periodic structure under the effect of folding and unfolding. In particular, we analyze the relationship between the periods of periodic sequences of the p-periodic difference equation and the periods of the corresponding subsequences related to the folded systems.

1. Introduction

Let $I = [a, b] \subset \mathbb{R}$ be a closed interval with $-\infty < a < b < \infty$, and denote by $\mathcal{C}(I)$ the space of continuous maps $f : I \to I$. Given a p-periodic sequence $\{f_n\}_{n \geqslant 0} \subset \mathcal{C}(I)$, that is, $f_{n+p} = f_n$ for all non-negative integers n, we consider the p-periodic difference equation

$$x_{n+1} = f_n(x_n), \quad n \in \mathbb{N} := \{0, 1, \ldots\}. \tag{1.1}$$

It is worth emphasizing here that we use period to denote the minimal period, unless mentioned otherwise. To stress the role of the maps involved in this p-periodic difference equation, we use the representation

$$[f_0, f_1, \ldots, f_{p-1}].$$

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If $p = 1$, then the equation is autonomous and we simply denote it by f_0, that is, the alternated system is reduced to the classical discrete system $x_{n+1} = f_0(x_n)$, $n \geqslant 0$. Periodic difference equations of the form given in Eq. (1.1) appear in a natural way in technical and social sciences related to processes involving two or more interactions. For the knowledge of the behavior of the general system, it is necessary to alternate different discrete dynamical systems corresponding to each period of the process. In this sense, it is interesting to stress that Eq. (1.1) can model certain populations in a periodically fluctuating environment [12,9,7,16,10]. For $p = 2$, we also find applications related to physics [1,13] and economy [14,15] in the context of Parrondo's paradox [11].

For an initial condition $x_0 \in I$, the solution or orbit through x_0 is given by

$$\begin{aligned} \mathcal{O}^+(x_0) &:= \{x_0, x_1, x_2, \ldots\} \\ &= \{x_0, f_0(x_0), f_1\big(f_0(x_0)\big), f_2\big(f_1\big(f_0(x_0)\big)\big), \ldots\}. \end{aligned} \tag{1.2}$$

Characterizing periodic solutions of Eq. (1.1) have been a topic of growing interest in the past decade [2–4,6,8]. An orbit $\mathcal{O}^+(x_0)$ is called r-cycle if r is the smallest positive integer for which $x_{n+r} = x_n$ for all $n \in \mathbb{N} := \{0, 1, 2, \ldots\}$. Notice that we use r-cycle rather than "r-periodic solution" to distinguish between talking about the periodicity of the system and periodicity of solutions. We also say r is the period or order of $\mathcal{O}^+(x_0) = (x_n)$, which can be denoted by $\mathrm{ord}_{[f_0,\ldots,f_{p-1}]}(x_0)$. By $\mathrm{P}([f_0, \ldots, f_{p-1}])$ and $\mathrm{Per}([f_0, \ldots, f_{p-1}])$ we denote the sets of periodic points and periods of $[f_0, \ldots, f_{p-1}]$, respectively. Note that in discrete autonomous systems if $x_0 \in \mathrm{P}([f_0])$, then $x_n \in \mathrm{P}([f_0])$ for all n, while in periodic nonautonomous systems [3], if $x_0 \in \mathrm{P}([f_0, \ldots, f_{p-1}])$ then $x_{nd} \in \mathrm{P}([f_0, \ldots, f_{p-1}])$, where d is the greatest common divisor between p and the order of (x_n). Throughout this paper, we use $\gcd(p, q)$ and $\mathrm{lcm}(p, q)$ to denote the greatest common divisor and least common multiple between p and q, respectively. Eq. (1.1) can be of minimal period p on the interval I, but reduces to an equation of shorter period on a nontrivial subinterval of I. In such a case, it is possible to treat Eq. (1.1) based on the new shorter period and the partitioned domain. We refer the reader to [2] for more information about this scenario. However, we consider this scenario to be a degenerate one and avoid it throughout this paper.

In this paper, we focus on the notion of folding certain maps of Eq. (1.1) to shorten its period, while the unfolding is used to denote the reversed process. For instance, suppose that we have a 6-periodic system $[f_0, f_1, \ldots, f_5]$. We can define the map $F := f_5 \circ f_4 \circ \cdots \circ f_0$, then deal with the autonomous equation $x_{n+1} = F(x_n)$. Define the maps $F_0 := f_1 \circ f_0$, $F_1 := f_3 \circ f_2$, $F_2 := f_5 \circ f_4$, then deal with the periodic alternating system $[F_0, F_1, F_2]$. Or we can define the maps $F_0 := f_2 \circ f_1 \circ f_0$ and $F_1 := f_5 \circ f_4 \circ f_3$, then deal with the periodic alternating system $[F_0, F_1]$. One of the main objectives here is to characterize the periodic structures in all these possible scenarios. The notion of folding and unfolding was introduced by Al-Salman and AlSharawi in [2]. However, we feel it has not blossomed yet; and therefore, we write this paper to develop further results that will be used by the authors in [5] to characterize forcing between cycles. It is worth mentioning here that the results of this paper are mostly based on the combinatorial structure of orbits, which do not need continuity. However, we assumed continuity of maps to keep the sittings within our long term goal which is the characterization of forcing between cycles. In Section 2, we discuss the notion of folding and its effect on periodic solutions. Given an r-cycle of Eq. (1.1), our main goal is to obtain information on the periods of the subsequences obtained by folding the initial system. Finally, the notion of unfolding and its concerning results are provided in Section 3. Here, from a q-cycle of a folded system we obtain information on the possible period of the corresponding unfolded cycle.

2. Folding in periodic difference equations

Consider the p-periodic equation in Eq. (1.1), and let k be a positive integer. For $j = 0, 1, \ldots$, define the maps