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Convergence rates in the law of large numbers for arrays of martingale differences

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ABSTRACT

We study the convergence rates in the law of large numbers for arrays of martingale differences. For $n \geqslant 1$, let $X_{n1}, X_{n2}, \ldots$ be a sequence of real valued martingale differences with respect to a filtration $\{\emptyset, \Omega\} = \mathcal{F}_{n0} \subset \mathcal{F}_{n1} \subset \mathcal{F}_{n2} \subset \cdots$, and set $S_{nn} = X_{n1} + \cdots + X_{nn}$. Under a simple moment condition on $\sum_{j=1}^{n} \mathbb{E}[|X_{nj}|^\gamma | \mathcal{F}_{n,j-1}]$ for some $\gamma \in (1,2]$, we show necessary and sufficient conditions for the convergence of the series $\sum_{n=1}^{\infty} \phi(n) P\{|S_{nn}| > \varepsilon n^\alpha\}$, where α, $\varepsilon > 0$ and ϕ is a positive function; we also give a criterion for $\phi(n) P\{|S_{nn}| > \varepsilon n^\alpha\} \to 0$. The most interesting case where ϕ is a regularly varying function is considered with attention. In the special case where $(X_{nj})_{j \geqslant 1}$ is the same sequence $(X_j)_{j \geqslant 1}$ of independent and identically distributed random variables, our result on the series $\sum_{n=1}^{\infty} \phi(n) P\{|S_{nn}| > \varepsilon n^\alpha\}$ corresponds to the theorems of Hsu, Robbins and Erdös (1947, 1949) if $\alpha = 1$ and $\phi(n) = 1$, of Spitzer (1956) if $\alpha = 1$ and $\phi(n) = 1/n$, and of Baum and Katz (1965) if $\alpha > 1/2$ and $\phi(n) = n^{b-1}$ with $b \geqslant 0$. In the single martingale case (where $X_{nj} = X_j$ for all n and j), it generalizes the results of Alsmeyer (1990). The consideration of martingale arrays (rather than a single martingale) makes the results very adapted in the study of weighted sums of identically distributed random variables, for which we prove new theorems about the rates of convergence in the law of large numbers. The results are established in a more general setting for sums of infinitely many martingale differences, say $S_{n,\infty} = \sum_{j=1}^{\infty} X_{nj}$ instead of S_{nn}. The obtained results improve and extend those of Ghosal and Chandra (1998). The one-sided cases and the supermartingale case are also considered.

1. Introduction

The convergence rates in the law of large numbers have been considered by many authors. Let $(X_j)_{j \geqslant 1}$ be a sequence of independent and identically distributed (i.i.d.) random variables defined on a probability

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space $(\Omega, \mathcal{F}, P)$ with $\mathbb{E}X_i = 0$ and set $S_n = \sum_{j=1}^n X_j$. By the law of large numbers, $P\{|S_n| > \varepsilon n\} \to 0$ for $\varepsilon > 0$. Hsu and Robbins [12] introduced the notion of complete convergence, and showed that

$$\sum_{n=1}^{\infty} P\{|S_n| > \varepsilon n\} < \infty \quad \text{for all } \varepsilon > 0 \tag{1.1}$$

if $\mathbb{E}X_1^2 < \infty$; Erdös [8] proved that the converse also holds. Spitzer [31] showed that

$$\sum_{n=1}^{\infty} n^{-1} P\{|S_n| > \varepsilon n\} < \infty \quad \text{for all } \varepsilon > 0 \tag{1.2}$$

whenever $\mathbb{E}X_1 = 0$. Katz [16] and Baum and Katz [4] proved that, for $p = \frac{1}{\alpha}$ and $\alpha \geqslant \frac{1}{2}$, or $p > \frac{1}{\alpha}$ and $\alpha > \frac{1}{2}$,

$$\sum_{n=1}^{\infty} n^{p\alpha-2} P\{|S_n| > \varepsilon n^{\alpha}\} < \infty \quad \text{for all } \varepsilon > 0 \tag{1.3}$$

if and only if $\mathbb{E}|X_1|^p < \infty$. Lai [22] studied the limiting case where $p > 2$ and $\alpha = \frac{1}{2}$. Gafurov and Slastnikov [9] considered the case where $(n^{p\alpha-2})$ and (n^{α}) are replaced by more general sequences. Chow and Lai [6] studied the corresponding one-sided analogues of the above results and proved that if $\mathbb{E}|X_1|^{\gamma} < \infty$ for some $1 \leqslant \gamma \leqslant 2$, then for $\alpha > \frac{1}{\gamma}$ and $p > \frac{1}{\alpha}$,

$$\sum_{n=1}^{\infty} n^{p\alpha-2} P\{S_n \geqslant \varepsilon n^{\alpha}\} < \infty \quad \text{for all } \varepsilon > 0 \tag{1.4}$$

if and only if $\mathbb{E}(X_1)_+^p < \infty$, where $(X_1)_+ = \max(0, X_1)$ denotes the positive part of X_1 (this notation will be used throughout the paper). Many authors have considered the generalizations of the theorem of Baum and Katz [4] to arrays of independent but not necessarily identically distributed random variables, see e.g. Li et al. [25], Hu et al. [14,15,13], Kuczmaszewska [20], Sung et al. [33], Kruglov et al. [19].

Let $(X_j)_{j\geqslant 1}$ be a sequence of real-valued martingale differences defined on a probability space $(\Omega, \mathcal{F}, P)$, adapted to a filtration $(\mathcal{F}_j)$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$. This means that for each (integer) $j \geqslant 1$, X_j is $\mathcal{F}_j$-measurable and $\mathbb{E}[X_j|\mathcal{F}_{j-1}] = 0$ a.s. A natural question is whether the pre-mentioned theorem of Baum and Katz [4] is still valid for martingale differences (X_j). Lesigne and Volný [24] proved that for $p \geqslant 2$, $\sup_{j\geqslant 1} \mathbb{E}|X_j|^p < \infty$ implies

$$P(|S_n| > \varepsilon n) = O(n^{-p/2}) \tag{1.5}$$

(as usual we write $a_n = o(b_n)$ if $\lim_{n\to\infty} a_n/b_n = 0$, and $a_n = O(b_n)$ if the sequence (a_n/b_n) is bounded), and that the exponent $p/2$ is the best possible, even for strictly stationary and ergodic sequences of martingale differences. Therefore the theorem of Baum and Katz does not hold for martingale differences without additional conditions. (Stoica [32] claimed that the theorem of Baum and Katz still holds for $p > 2$ in the case of martingale differences without additional assumption; but his claim is a contradiction with the conclusion of Lesigne and Volný [24], and his proof contains an error: when $p > 2$, one cannot choose α satisfying (6) of [32].) Alsmeyer [1] proved that the theorem of Baum and Katz for $p > \frac{1}{\alpha}$ and $\frac{1}{2} < \alpha \leqslant 1$ still holds for martingale differences $(X_j)_{j\geqslant 1}$ if for some $\gamma \in (1/\alpha, 2]$ and $q \in [1, \infty]$ with $q > (p\alpha - 1)/(\gamma\alpha - 1)$,

$$\sup_{n\geqslant 1} \left\| \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}[|X_j|^{\gamma}|\mathcal{F}_{j-1}] \right\|_q < \infty, \tag{1.6}$$