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Inequalities for two type potential operators on differential forms

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ABSTRACT

The purpose of this paper is to derive some strong-type inequalities for convolution type potential operator applied on differential forms. Caccioppoli-type inequalities for integral type potential operator acting on A-harmonic tensor are also obtained.

1. Introduction

In this paper, we mainly consider the following convolution type potential operator acting on differential forms and develop some norm inequalities for the potential operator.

Given a nonnegative, locally integrable function Φ, the potential operator T_Φ is defined by a convolution integral

$$T_\Phi u(x) = \sum_I \left(\int_{\mathbb{R}^n} \Phi(x-y) u_I(y)\, dy \right) dx_I \tag{1.1}$$

provided this integral exists for almost all $x \in \mathbb{R}^n$, where $u(x)$ is a k-form defined on $\mathbb{R}^n$ and the summation is over all ordered k-tuples I.

The function Φ is also assumed to be a wide class of kernels satisfying the following weak growth condition (D):

There are constants $\delta, c > 0$, and $0 \leqslant \varepsilon < 1$ with the property that

$$\sup_{2^k < |x| < 2^{k+1}} \Phi(x) \leqslant \frac{c}{2^{kn}} \int_{\delta(1-\varepsilon)2^k < |y| < 2\delta(1+\varepsilon)2^k} \Phi(y)\, dy$$

for all $k \in \mathbb{Z}$.

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Differential forms can be viewed as an extension of functions. When $u(x)$ is a 0-form, the above-mentioned notations are in accord with that of function spaces, and the potential operator T_ϕ we studied in this paper degenerates into the operator which C. Pérez discussed in [7]. Namely, for any Lebesgue measurable function f, T_Φ is given as follows

$$T_\Phi f(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy.$$

This degenerated operator which was also introduced by R. Kerman and E.T. Sawyer in [3] plays a critical role in the proof of trace type inequalities. These inequalities can be used to derive eigenvalue estimates for Schrödinger operators which are closely relevant to quantum mechanics.

It is obvious that T_Φ includes the fractional integral operator I_α when the kernel $\Phi(t) = |t|^{\alpha-n}$, $0 < \alpha < n$. Other than fractional integral operator I_α, another important example for the convolution integral is Bessel potential $J_{\beta,\lambda}$ with kernel $K_{\beta,\lambda}$ defined by its Fourier transform $\widehat{K_{\beta,\lambda}}(\xi) = (\lambda^2 + |\xi|^2)^{\frac{\beta}{2}}$, $\beta, \lambda > 0$. Thus it is reasonable to assume the kernels Φ are radially decreasing functions from the two cases. Meanwhile C. Pérez pointed out that the weak growth condition (D) introduced before is more general in his paper [7].

Recently, many intensive and significant conclusions have been drawn with regard to the above-mentioned operators. S. Samko and B. Vakulov proved a weighted Sobolev-type theorem for the Riesz fractional integral operator I_α in the weighted Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n, \rho)$ with the variable exponent $p(x)$ in [9]. As to the weighted case for I_α and M_α, B. Muckenhoupt and R.L. Wheeden obtained norm comparison for the two operators in [4], which had applications in nonlinear potentials especially in the proof of the Hedberg–Wolff theorem [1].

In the usual Euclidean space $\mathbb{R}^n$, if $0 < \alpha < n$, Riesz fractional integral operator can be defined by the kernel $\Phi(t) = |t|^{\alpha-n}$ as follows

$$I_\alpha f(x) = \int_{\mathbb{R}^n} f(y) \frac{1}{|x-y|^{n-\alpha}}\, dy.$$

The corresponding maximal operator is denoted by

$$M_\alpha f(x) = \sup_{B:\, x\in B} r(B)^{\alpha-n} \int_B \bigl|f(y)\bigr|\, dy,$$

where B is a Euclidean ball and $r(B)$ is the radius of B.

Also in the study of Riesz potential theory and partial differential equations, it's necessary to obtain the integrability for fractional integral operator, potential operator and corresponding maximal operators in various measure spaces.

As presented in the above paragraphs, convolution type potential operator is a sort of general operator, when the kernel function takes some special functions or satisfies certain conditions, it includes many classic operators such as fractional integral operator, Calderón–Zygmund operator and commutator.

Besides this type potential operator, we simply consider the following integral type potential operator defined by

$$T_K u(x) = \sum_I \biggl(\int_E K(x,y) u_I(y)\, dy \biggr) dx_I, \tag{1.2}$$

where $K(x,y)$ is a nonnegative measurable function, $E \in \Omega$ is a bounded open subset of $\mathbb{R}^n$, and $u(x)$ is an A-harmonic tensor given below. Caccioppoli-type inequalities will be developed for this operator.