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A mathematical aspect of a tunnel-junction for spintronic qubit

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ABSTRACT

We consider the Dirac particle that lives in the 1-dimensional configuration space consisting of two quantum wires and a junction between the two. We regard the spin of a Dirac particle as spintronic qubit. We give concrete formulae explicitly expressing the one-to-one correspondence between every self-adjoint extension of the minimal Dirac operator and its corresponding boundary condition of the wave functions of the Dirac particle. We then show that all the boundary conditions can be classified into just two types. The two types are characterized by whether the electron passes through the junction or not. We also show how the tunneling produces its own phase factor and what is the relation between the phase factor and the spintronic qubit in the tunneling boundary condition.

1. Introduction

In this paper we pay attention to electron spin as spintronic qubit from a mathematical point of view. The current cutting-edge technology has been developed seeking the realization of quantum information and quantum computation. One of the candidates for qubit is the electron spin [12,22], that is, the *spintronic qubit* (i.e., electron-spin qubit) called in spintronics [3,10,19,21,22]. It is remarkable that the transportation of a single electron as a qubit has been demonstrated experimentally [15,23] as well as its manipulation in a semiconductor quantum dot [19,26] and in a semiconductor nanowire [24]. Thus, this paper mathematically deals with the Dirac particle living in a configuration space. Our configuration space consists of two quantum wires and a junction. Although we actually have to determine a concrete physical object for the junction, we consider the junction as a black box so that it has mathematical arbitrariness. We regard the wires as the union of the intervals, $(-\infty, -\Lambda) \cup (\Lambda, \infty)$ with $\Lambda > 0$, for mathematical simplicity. Namely, the segment $[-\Lambda, \Lambda]$ with length 2Λ plays a role of the junction. Many (mathematical) physicists have investigated the individual expressions for various boundary conditions of the wave functions of the Dirac particle for the corresponding self-adjoint extension of the Dirac operator in the case of the point interaction (i.e., $\Lambda = 0$) [1,2,7,13,18,28]. Meanwhile, the Dirac operator consists of the combination of the Dirac matrices and the momentum operator of electron. The boundary conditions of the self-adjoint extensions of the momentum

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operator have been studied as well as the Dirac operator in mathematics [8,16,25]. It is worth noting that there is a theory of boundary triplet in mathematics generally to handle the boundary condition, and the theory has still been developed [4–6,8,9,11].

For the Schrödinger particle under the same mathematical setup as ours, we showed in [14,17,29] that some self-adjoint extensions require the appearance of a phase factor in their own boundary condition when the wave functions pass through the junction. As for our Dirac particle, in the case where $\Lambda = 0$, we can find the appearance of such a phase factor in Benvegnù and Dąbrowski's four-parameter family called in (mathematical) physics (see (15) of [7]). Their four-parameter family gives an impact on the spin as well. In this paper we will go ahead and make an in-depth mathematical observation for all self-adjoint extensions of our minimal Dirac operator. We follow the machinery in [14,16,17,25,29] based on the von Neumann's theory [27,30], in which all self-adjoint extensions of the minimal Dirac operator are parameterized by $U \in U(2)$, where $U(2)$ is unitary group of degree 2. We will give concrete formulae explicitly expressing the one-to-one correspondence between every boundary condition of the wave functions of our Dirac particle and its corresponding self-adjoint extension of the minimal Dirac operator (see Theorem 4.4 and Proposition 4.6).

From a physical point of view, it is natural and essential to classify the types of boundary conditions according to whether the type is for a transparent boundary or a non-transparent one as in [4,20], in which the boundary conditions of the wave functions of the Schrödinger particle are argued in the light of scattering theory. We will press ahead with their physical view. We will then prove that for our Dirac particle all the boundary conditions are completely classified into two types (see Corollary 4.5). One of them is the type which states that the wave functions do not pass through the junction and make the perfect reflection at $-\Lambda$ and at $+\Lambda$, and the other is the type which states that the wave functions do pass through the junction (see Theorems 4.2 and 4.4). The latter type is equivalent to Benvegnù and Dąbrowski's four-parameter family (see Proposition 4.7). Thus, we can realize the concrete relation between the phase factor and the spintronic qubit. The both types can be characterized by three parameters, $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{C}$ with $|\gamma_1|^2 + |\gamma_2|^2 = |\gamma_3| = 1$. The parameters γ_1 and γ_2 respectively govern the reflection of the wave function at the boundaries and its tunneling through the junction. We will then find that the criterion determining the type is whether γ_2 is zero or not. By applying our formulae, we will eventually clarify when the self-adjoint extension requires the appearance of the special phase factor $e^{i\theta}$ in the boundary condition, and moreover, show how the tunneling parameter γ_2 determines the phase θ (Corollary 4.9), which convinces us that this θ is the phase peculiar to the tunneling of the wave functions. Therefore, we call θ the *tunneling phase*.

2. Mathematical notations and notions

We denote by $\langle\,|\,\rangle_{\mathcal{H}}$ the inner product of a separable Hilbert space $\mathcal{H}$, where we suppose that the right hand side of the inner product $\langle\,|\,\rangle_{\mathcal{H}}$ is linear. We here prepare some notations and notions used in operator theory.

Let $\mathcal{L}(\mathcal{H})$ denote the set of all (linear) operators acting in $\mathcal{H}$. We always omit the word of 'linear' from the notion of linear operator because we consider only linear operators in this paper. We denote by $D(A)$ the domain of every operator $A \in \mathcal{L}(\mathcal{H})$. We denote by $A \subset B$ or $B \supset A$ an extension B of the operator A. For every operator $A \in \mathcal{L}(\mathcal{H})$ and subspace $S \subset \mathcal{H}$ with $S \subset D(A)$, $A \lceil S \in \mathcal{L}(\mathcal{H})$ means the restriction of A on S. Let $I_{\mathcal{H}}$ or I denote the identity operator on $\mathcal{H}$.

Let $A_0 \in \mathcal{L}(\mathcal{H})$ be closed symmetric now. An operator $A \in \mathcal{L}(\mathcal{H})$ is said to be a *self-adjoint extension* of A_0, provided that the condition, $A_0 \subset A$, holds and A is self-adjoint. For every closed symmetric operator $A_0 \in \mathcal{L}(\mathcal{H})$, we respectively define *deficiency subspaces* $\mathcal{K}_+(A_0)$ and $\mathcal{K}_-(A_0)$ by $\mathcal{K}_\pm(A_0) := \ker(\pm i - A_0^*)$, and moreover, *deficiency indices* $n_+(A_0)$ and $n_-(A_0)$ by $n_\pm(A_0) := \dim \mathcal{K}_\pm(A_0)$.

Our argument in this paper is based on the following proposition: