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Some Liouville-type theorems for harmonic functions on Finsler manifolds ☆

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ABSTRACT

In this paper, we give some Liouville-type theorems for L^p ($p \in \mathbb{R}$) harmonic (resp. subharmonic, superharmonic) functions on forward complete Finsler manifolds. Moreover, we derive a gradient estimate for harmonic functions on a closed Finsler manifold. As an application, one obtains that any harmonic function on a closed Finsler manifold with nonnegative weighted Ricci curvature Ric_N ($N \in (n, \infty)$) must be constant.

1. Introduction

The classical Liouville theorem states that every nonnegative (or bounded) harmonic function on $\mathbb{R}^n$ must be constant. Since the 1970s, various Liouville theorems for harmonic (subharmonic) functions have been extensively studied on complete Riemannian manifolds. For example, in 1975 and 1976, S.-T. Yau proved that every positive (or bounded) harmonic function on a complete Riemannian manifold with nonnegative Ricci curvature must be constant and there are no nonnegative L^p subharmonic functions and no L^p-harmonic functions for any $1 < p < \infty$ on a complete noncompact Riemannian manifold [21,22]. In 1994, Sturm extended Yau's results for $p \in \mathbb{R}$ and $p \neq 1$ [16]. Moreover, L^1-Liouville theorems for harmonic functions on a complete Riemannian manifold had been studied by P. Li and R. Schoen, etc. ([5,7] and references therein). It is known that these Liouville theorems play an important role in analyzing the underlying manifold. Recently, X. Li generalized the above various Liouville theorems for Laplacian operators to those for general symmetric diffusion operators on complete Riemannian manifolds and gave some applications [6].

A natural question is how to generalize the above Liouville theorems on Riemannian manifolds to Finsler manifolds. In this note, we study this question. Let (M, F) be an n-dimensional *Finsler manifold*, which

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means a connected smooth manifold equipped with a Finsler structure (or metric) $F : TM \to [0, +\infty)$. For any weakly differentiable function u on (M, F), define the gradient vector of u, denoted by ∇u, the dual of the 1-form du under the Legendre transform. In general, the gradient vector field ∇u is not differentiable at points with $\nabla u(x) = 0$ even if (M, F) and u are smooth. However it is continuous on M. To consider the global analysis on a Finsler manifold M, throughout the paper we always assume that (M, F) is orientable.

Let (M, F, m) be a Finsler manifold with a smooth volume measure m. Set $dm = \sigma(x)\,dx$. For a weakly differentiable vector field $V : M \to TM$, we define its *divergence* $\mathrm{div}_m V : M \to R$ through the identity

$$\int_M \phi \,\mathrm{div}_m V \,dm = -\int_M d\phi(V)\,dm, \tag{1.1}$$

where $\phi \in C_c^\infty(M)$. With these preparations, we define the *Finsler Laplacian* Δ_m acting on functions $u \in W^{1,2}(M)$ formally by $\Delta_m u := \mathrm{div}_m(\nabla u)$. To be more precise, $\Delta_m u$ is the distributional Laplacian defined through the identity

$$\int_M \phi \Delta_m u \,dm = -\int_M d\phi(\nabla u)\,dm, \tag{1.2}$$

for all $\phi \in C_c^\infty(M)$ [14,15]. For the sake of convenience, in the following we simply denote Δ_m as Δ.

It is worth mentioning that there are various definitions for the Laplacian in Finsler geometry, which were introduced respectively by Bao and Lacky [1], Centore [2], Shen [14], Mo [9], Thomas [19], etc. since the volume measure on M is not unique in Finsler geometry. In fact, these operators defined in [1,2,9,19] are essentially weighted Laplacian operators on a weighted Riemannian manifold $(M, \tilde{g}, d\mu)$, which rely on some Riemannian metric $\tilde{g}$ or volume measure on an underlying manifold induced by the pull-back of the Sasakian metric from the slit tangent bundle $TM \setminus \{0\}$ to SM. Moreover, they are linear elliptic operators. In this paper, we use the more natural definition of the Laplacian introduced by Z. Shen in [14], which is from the critical value of the energy functional. It is a nonlinear elliptic operator on an underlying manifold with respect to any measure. All Finsler Laplacian operators mentioned here are reduced to the usual Laplace operator on a Riemannian manifold if F is a Riemannian metric.

Given $u \in C^2(M)$, by (1.2), the Finsler Laplacian is locally expressed by

$$\Delta u(x) = \mathrm{div}_m\bigl(\nabla u(x)\bigr) = \frac{1}{\sigma(x)}\frac{\partial}{\partial x^i}\left(\sigma(x) g^{ij}\bigl(x, \nabla u(x)\bigr)\frac{\partial u}{\partial x^j}\right), \tag{1.3}$$

where $x \in M_u := \{x \in M \mid du(x) \neq 0\}$, $g_{ij}(x, y) := \frac{1}{2}[F^2(x, y)]_{y^i y^j}$ is the fundamental tensor of F and $[g^{ij}(x, y)] := [g_{ij}(x, y)]^{-1}$. We say a function $u \in C^2(M)$ is a *harmonic* (resp. *subharmonic*, *superharmonic*) function on M if $\Delta u = (\geqslant, \leqslant)0$ in a weak sense.

In this paper, the main purpose is to study some Liouville properties of harmonic (resp. subharmonic, superharmonic) functions on a Finsler manifold. To state some results, we need to introduce some notation.

Let

$$\Lambda := \sup_{(x,y) \in TM \setminus \{0\}} \left\{ \frac{F(x, -y)}{F(x, y)} \right\}, \tag{1.4}$$

which is called the *reversibility* of F. We say the reversibility of F is finite if $\Lambda < \infty$. In particular, if F is a Riemannian metric, then $\Lambda = 1$.

Fix a point $x_0 \in M$, for $r > 0$, we denote a forward geodesic ball of radius r with center at x_0 by $B_r^+(x_0)$ and the volume of $B_r^+(x_0)$ with respect to the measure m by $V(r)$. Let