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On strongly monotone solutions of a class of cyclic systems of nonlinear differential equations

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The n-dimensional cyclic systems of first order nonlinear differential equations

$$x_i' + p_i(t)x_{i+1}^{\alpha_i} = 0, \quad i = 1, \dots, n \quad (x_{n+1} = x_1), \tag{A}$$

$$x_i' = p_i(t)x_{i+1}^{\alpha_i}, \quad i = 1, \dots, n \quad (x_{n+1} = x_1), \tag{B}$$

are analyzed in the framework of regular variation. Under the assumption that $\alpha_1 \cdots \alpha_n < 1$ and $p_i(t)$, $i = 1, \dots, n$, are regularly varying functions, it is shown that the situation in which system (A) (resp. (B)) possesses decreasing (resp. increasing) regularly varying solutions of negative (resp. positive) indices can be completely characterized, and moreover that the asymptotic behavior of such solutions is governed by the unique formula describing their order of decay (resp. growth) precisely. Examples are presented to demonstrate that the main results for (A) and (B) can be applied effectively to some higher order scalar nonlinear differential equations to provide new accurate information about the existence and the asymptotic behavior of their positive strongly monotone solutions.

1. Introduction

We consider nonlinear cyclic systems of differential equations of the forms

$$x_i' + p_i(t)x_{i+1}^{\alpha_i} = 0, \quad i = 1, 2, \dots, n, \qquad x_{n+1} = x_1, \tag{A}$$

and

$$x_i' = p_i(t)x_{i+1}^{\alpha_i}, \quad i = 1, 2, \dots, n, \qquad x_{n+1} = x_1, \tag{B}$$

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under the assumptions that

(a) α_i, $i = 1, 2, \dots, n$, are positive constants such that $\alpha_1\alpha_2 \cdots \alpha_n < 1$;
(b) $p_i : [a, \infty) \to (0, \infty)$, $i = 1, 2, \dots, n$, are continuous functions.

By a positive solution of (A) (resp. (B)) we mean a continuously differentiable vector function $(x_1(t), x_2(t), \dots, x_n(t))$ all of whose components $x_i(t)$ are defined and positive in a neighborhood of infinity and satisfy the system (A) (resp. (B)) there. We are particularly interested in *strongly monotone solutions* of (A) and (B) as defined below.

(i) A positive solution $(x_1(t), x_2(t), \dots, x_n(t))$ of (A) is called *strongly decreasing* if $x_i(t) \to 0$ as $t \to \infty$ for $i = 1, 2, \dots, n$. Such a solution satisfies the system of integral equations

$$x_i(t) = \int_t^\infty p_i(s) x_{i+1}(s)^{\alpha_i}\, ds, \quad i = 1, 2, \dots, n, \tag{1.1}$$

for all large t.

(ii) A positive solution $(x_1(t), x_2(t), \dots, x_n(t))$ of (B) is called *strongly increasing* if $x_i(t) \to \infty$ as $t \to \infty$ for $i = 1, 2, \dots, n$. Such a solution satisfies the system of integral equations

$$x_i(t) = x_{i,0} + \int_T^t p_i(s) x_{i+1}(s)^{\alpha_i}\, ds, \quad t \geqslant T, \quad i = 1, 2, \dots, n, \tag{1.2}$$

for some constants $T > a$ and $x_{i,0} > 0$, $i = 1, 2, \dots, n$.

The aim of this paper is to acquire as detailed and precise information as possible about the existence and asymptotic behavior of strongly monotone solutions of (A) and (B) and to utilize the information thus obtained to study positive solutions of scalar higher order nonlinear differential equations such as

$$\left(p(t)\left|x^{(n)}\right|^{\alpha-1} x^{(n)}\right)^{(n)} = q(t)|x|^{\beta-1} x. \tag{1.3}$$

Since the problem under study is very difficult to solve for systems (A) and (B) with general continuous $p_i(t)$, we limit ourselves to the case where the coefficients $p_i(t)$ are *regularly varying* in the sense of Karamata (for the definition see Section 2) and focus our attention on *regularly varying solutions* of systems (A) and (B). Asymptotic analysis of positive solutions of differential equations in the framework of regular variation has attracted increasing interest of various authors since the publication of the book of Marić [13]. See, for example, the papers [5,6,8–11]. The study in the same spirit of systems of differential equations has been attempted by the present authors [3,4] for two-dimensional prototypes of (A) and (B). The present work is designed to examine the possibility of generalizing these two-dimensional results to multi-dimensional cyclic systems of the forms (A) and (B).

The main body of the paper is divided into Part I and Part II which are devoted to the study of strongly monotone solutions of systems (A) and (B), respectively. In each part analysis with the help of basic theory of regular variation is carried out to obtain thorough and complete knowledge of all strongly monotone regularly varying solutions with positive or negative indices for the cyclic systems under consideration. Furthermore it will be shown that our main results on systems (A) and (B) have applications to higher order scalar differential equations including (1.3), providing new nontrivial information about the structure of their strongly monotone solutions.