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Reflected mean-field backward stochastic differential equations. Approximation and associated nonlinear PDEs

Juan Li [1]

School of Mathematics and Statistics, Shandong University (Weihai), Weihai 264209, PR China

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ABSTRACT

Mathematical mean-field approaches have been used in many fields, not only in Physics and Chemistry, but also recently in Finance, Economics, and Game Theory. In this paper we will study a new special mean-field problem in a purely probabilistic method, to characterize its limit which is the solution of mean-field backward stochastic differential equations (BSDEs) with reflections. On the other hand, we will prove that this type of reflected mean-field BSDEs can also be obtained as the limit equation of the mean-field BSDEs by penalization method. Finally, we give the probabilistic interpretation of the nonlinear and nonlocal partial differential equations with the obstacles by the solutions of reflected mean-field BSDEs.

1. Introduction

Mathematical mean-field approaches have been used in many fields. To work on a stochastic limit approach to a mean-field problem is inspired at the one hand by classical mean-field approaches in Statistical Mechanics and Physics, by similar methods in Quantum Mechanics and Quantum Chemistry, but also by a recent series of papers by Lasry and Lions (see [16] and the references inside cited) who studied mean-field games. And also it has been strongly inspired by the McKean–Vlasov partial differential equations (PDEs) which have found a great interest in the last years and have been studied with the help of stochastic methods by many authors. On the other hand, in the last years models of large stochastic particle systems with mean-field interaction have been studied by many authors; they have described them by characterizing their asymptotic behavior when the size of the system becomes very large, and also have shown that probabilistic methods allow to study the solution of linear McKean–Vlasov PDE. The reader is referred, for example, to the works by Borkar and Kumar [4], Bossy [5], Bossy and Talay [6], Chan [11], Kotelenez [15], McKean [19], Méléard [20], Overbeck [21], Pra and Hollander [24], Sznitman [26,27], Talay and Vaillant [28], and all the references therein. More details may refer to Buckdahn, Djehiche, Li and Peng [9] and the references inside cited.

Buckdahn, Djehiche, Li and Peng [9] studied a special mean-field problem in a purely stochastic approach. They considered a stochastic differential equation that describes the dynamics of a particle $X^{(N)}$ influenced by the dynamics of N other particles, which are supposed to be independent identically distributed and of the same law as $X^{(N)}$. This equation (of rank N) is then associated with a backward stochastic differential equation (BSDE). After having proven the existence and the uniqueness of a solution $(X^{(N)}, Y^{(N)}, Z^{(N)})$ for this couple of equations the authors of [9] investigated its limit behavior. With a new approach which uses the tightness of the laws of the above sequence of triplets in a suitable space, and combines it with BSDE methods and the Law of Large Numbers, it was shown that $(X^{(N)}, Y^{(N)}, Z^{(N)})$ converges in L^2 to

E-mail address: juanli@sdu.edu.cn.

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the unique solution of a limit equation formed by a McKean–Vlasov stochastic differential equation and a mean-field backward stochastic differential equation. Furthermore, Buckdahn, Li and Peng [8] proved the existence and the uniqueness of the solution of mean-field BSDEs under the classical assumptions, the comparison theorem of mean-field BSDEs and gave a stochastic interpretation to McKean–Vlasov partial differential equations (PDEs) with the help of the solutions of mean-field BSDEs. Since then we want to work on another new special mean-field problem to get a new limit equation which is like reflected BSDE in some sense. On the other hand, since the works [8] and [9] on the mean-field BSDEs, there are many works on its generalizations, e.g., Wang [29] studied backward doubly SDEs of mean-field type and its applications; Shi, Wang and Yong [25] studied backward stochastic Volterra integral equations of mean-field type; Li and Luo [18] studied reflected BSDEs of mean-field type, they proved the existence and the uniqueness for reflected mean-field BSDEs; and also its applications, e.g., Andersson and Djehiche [1], Bensoussan, Sung, Yam and Yung [3], Buckdahn, Djehiche and Li [10], Li [17], Yong [31]. Reflected BSDEs were introduced by El Karoui, Kapoudjian, Pardoux, Peng and Quenez [13] in 1997. Later the theory of RBSDEs develops very quickly, because of its many applications, for example, in partial differential equations, finance and so on. More details may refer to Buckdahn and Li [7] and the references inside cited.

In this paper we will study another new special mean-field problem, and get its limit which is a new type of reflected BSDEs, we call it reflected mean-field BSDEs. Our objective here is to characterize such an equation, at one hand, as the limit of classical BSDEs with reflection and, on the other hand, as the limit of mean-field BSDEs with a penalization approach. The approximating reflected BSDEs (N) are discussed, and with an example it is in particular shown that these reflected BSDEs (N) don't obey the comparison principle. Furthermore, under an additional monotonicity assumption of the driving coefficient, the description of reflected mean-field BSDEs as monotonic limit of mean-field BSDEs without reflection is used to give through them a stochastic interpretation of associated nonlocal PDEs with obstacles. We show that the solution of the reflected mean-field BSDE is the unique viscosity solution of the associated nonlocal PDE with obstacles.

More precisely, we consider the following mean-field BSDE with reflections:

$$\begin{aligned}
&\text{(i)}\quad Y\in S^2_{\mathbf{F}}\big([0,T]\big),\qquad Z\in L^2_{\mathbf{F}}\big([0,T];\mathbb{R}^d\big)\quad\text{and}\quad K\in A^{2,c}_{\mathbf{F}}\big([0,T]\big);\\
&\text{(ii)}\quad Y_t=E\big[\Phi(x,X_T)\big]_{x=X_T}+\int_t^T E\big[g(s,\mathbf{u},\Lambda_s)\big]_{|\mathbf{u}=\Lambda_s}ds+K_T-K_t-\int_t^T Z_s\,dW_s;\\
&\text{(iii)}\quad Y_t\geqslant h(t,X_t),\quad \text{a.s., for all } t\in[0,T];\\
&\text{(iv)}\quad \int_0^T\big(Y_t-h(t,X_t)\big)\,dK_t=0,
\end{aligned}\tag{1.1}$$

where we have used the notation $\Lambda=(X,Y,Z)$; $T>0$ is a given finite time horizon; $W=(W_t)_{t\geqslant 0}$ is a d-dimensional Brownian motion; $X=(X_t)_{t\in[0,T]}$ is a driving n-dimensional adapted stochastic process.

Such type of mean-field BSDEs without reflections have been studied by Buckdahn, Li and Peng [8], they proved that such a mean-field BSDE gave a stochastic interpretation to the related nonlocal PDEs. In this paper we first prove that, under our standard assumptions the mean-field BSDE (1.1) with reflections will be the limit equation of the following reflected BSDE (N):

$$\begin{aligned}
&\text{(i)}\quad Y^N\in S^2_{\mathbf{F}}\big([0,T]\big),\qquad Z^N\in L^2_{\mathbf{F}}\big([0,T];\mathbb{R}^d\big)\quad\text{and}\quad K^N\in A^{2,c}_{\mathbf{F}}\big([0,T]\big);\\
&\text{(ii)}\quad Y^N_t=\xi^N+\int_t^T f^N\big(s,\Theta_N(Y_s,Z_s)\big)\,ds+K^N_T-K^N_t-\int_t^T Z^N_s\,dW_s,\quad t\in[0,T];\\
&\text{(iii)}\quad Y^N_t\geqslant L^N_t,\quad \text{a.s., for any } t\in[0,T];\\
&\text{(iv)}\quad \int_0^T\big(Y^N_t-L^N_t\big)\,dK^N_t=0,
\end{aligned}\tag{1.2}$$

where for $N\geqslant 1$ and $\omega\in\Omega$,

$$\begin{aligned}
&\xi^N(\omega):=\frac{1}{N}\sum_{k=1}^N\Phi\big(\Theta^k(\omega),X^N_T(\omega),X^N_T\big(\Theta^k(\omega)\big)\big),\\
&f^N(\omega,t,\mathbf{y},\mathbf{z}):=\frac{1}{N}\sum_{k=1}^N g\big(\Theta^k(\omega),t,X^N_t(\omega),(y_0,z_0),X^N_t\big(\Theta^k(\omega)\big),(y_k,z_k)\big),\\
&\qquad\text{for } t\in[0,T],\ \mathbf{y}=(y_0,\dots,y_N)\in\mathbb{R}^{N+1},\ \mathbf{z}=(z_0,\dots,z_N)\in\mathbb{R}^{(N+1)\times d},\\
&L^N_t(\omega):=h\big(\omega,t,X^N_t(\omega)\big),\quad t\in[0,T],
\end{aligned}$$