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On the reducibility of a class of finitely differentiable quasi-periodic linear systems ☆

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ABSTRACT

In this paper, we consider the following system

$$\dot{x} = \big(A + \varepsilon \widetilde{Q}(t)\big)x,$$

where A is a constant matrix with different eigenvalues, and $\widetilde{Q}(t)$ is quasi-periodic with frequencies $\omega_1, \omega_2, \ldots, \omega_r$. Moreover, $Q(\theta) = Q(\omega t) = \widetilde{Q}(t)$ has continuous partial derivatives $\frac{\partial^b Q}{\partial \theta_j^b}$ for $j = 1, 2, \ldots, r$, where $b > \frac{9}{4}r + 1 \in Z$, and the moduli of continuity of $\frac{\partial^b Q}{\partial \theta_j^b}$ satisfy a condition of finiteness (condition on an integral), which is more general than a Hölder condition. Under suitable hypothesis of non-resonance conditions and non-degeneracy conditions, we prove that for most sufficiently small ε, the system can be reducible to a constant coefficient differentiable equation by means of a quasi-periodic homeomorphism.

1. Introduction

Before stating our problem, we give some definitions and notations.

Definition 1.1. A function f is called a quasi-periodic function with frequencies $\omega = (\omega_1, \omega_2, \ldots, \omega_r)$ if $f(t) = F(\omega_1 t, \omega_2 t, \ldots, \omega_r t)$, where $F(\theta_1, \theta_2, \ldots, \theta_r)$ is 2π-periodic in all arguments and $\theta_i = \omega_i t$, $i = 1, 2, \ldots, r$. If $F(\theta)$ $(\theta = (\theta_1, \theta_2, \ldots, \theta_r))$ is analytic on $D_\rho = \{\theta \in C^r \mid |Im\theta_i| \leqslant \rho,\ i = 1, 2, \ldots, r\}$, we call $f(t)$ analytic quasi-periodic on D_ρ. Denote the sup-norm of f on D_ρ by $\|f\|_\rho = \sup_{\theta \in D_\rho} |F(\theta)|$.

Definition 1.2. A matrix function $Q(t) = (q_{ij}(t))_{1\leqslant i,j\leqslant n}$ is called analytic quasi-periodic on D_ρ if all $q_{ij}(t)$ $(i, j = 1, 2, \ldots, n)$ are analytic quasi-periodic on D_ρ.

Define the norm of Q on D_ρ by $\|Q\|_\rho = n \times \max_{1\leqslant i,j\leqslant n} \|q_{ij}\|_\rho$. Clearly, $\|Q_1 Q_2\|_\rho \leqslant \|Q_1\|_\rho \|Q_2\|_\rho$. For convenience, if Q is a constant matrix, we denote $\|Q\| = \|Q\|_\rho$. The average of Q is denoted by $[Q] = ([q_{ij}])_{1\leqslant i,j\leqslant n}$, where

$$[q_{ij}] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} q_{ij}(t)\, dt.$$

For the existence of its limit, see [2].

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Definition 1.3. If $f(x): R^r \to R$ is continuous in $x=(x_1,x_2,\ldots,x_r)$, we define $\|f(x)\|_M = \sup_{x\in M}|f(x)|$ for all functions f defined and bounded on some set $M \subseteq R^r$. Moreover, if a matrix function $Q(x)=(q_{ij}(x))_{1\leqslant i,j\leqslant n}$ is continuous in $x=(x_1,x_2,\ldots,x_r)$, we define $\|Q(x)\|_M = n\times \max_{1\leqslant i,j\leqslant n}\|q_{ij}(x)\|_M$ for $Q=(q_{ij})_{1\leqslant i,j\leqslant n}$ defined and bounded on some set $M\subseteq R^r$.

Problems. We consider the linear system $\dot{x}=A(t)x$, $x\in R^n$, where $A(t)$ is an $n\times n$ matrix. The well-known Floquent theorem tells us that if $A(t)$ is a T-periodic matrix, then the linear differential equation $\dot{x}=A(t)x$ is reducible to the constant coefficient differential equation $\dot{x}=Bx$ by a T-periodic change of variables. For the quasi-periodic coefficient system, R.A. Johnson and G.R. Sell [8] proved that if the quasi-periodic coefficients matrix $A(t)$ satisfies "full spectrum" conditions, then $\dot{x}=A(t)x$ is reducible. That is, there exists a quasi-periodic non-singular transformation $x=\phi(t)y$, where $\phi(t)$ and $\phi(t)^{-1}$ are quasi-periodic and bounded, such that $\dot{x}=A(t)x$ is transformed to $\dot{y}=By$, where B is a constant matrix. In [9], Jorba and Simó considered the reducibility of the following linear quasi-periodic system

$$\dot{x}=\big(A+\varepsilon Q(t)\big)x,\quad x\in R^n, \tag{1.1}$$

where A is a constant matrix with different eigenvalues. They proved that under the non-resonance conditions and the non-degeneracy conditions, there exists a non-empty Cantor subset E, such that for $\varepsilon\in E$, the system (1.1) is reducible. In [15], J. Xu considered the linear quasi-periodic system

$$\dot{x}=\big(A+\varepsilon Q(t)\big)x,\quad x\in R^n, \tag{1.2}$$

where A is a constant matrix with multiple eigenvalues. He proved that under the non-resonance conditions and the non-degeneracy conditions, there exists a non-empty Cantor subset E, such that the system (1.2) is reducible for $\varepsilon\in E$.

In the paper [6], to study one-dimensional linear Schrödinger equation

$$\frac{d^2q}{dt^2}+Q(\omega t)q=Eq,$$

Eliasson considered the following equivalent two-dimensional quasi-periodic Hamiltonian system:

$$\dot{p}=\big(E-Q(\omega t)\big)q,\qquad \dot{q}=p, \tag{1.3}$$

where Q is an analytic quasi-periodic function and E is an energy parameter. The result in [6] implies that for almost every sufficiently large E, the quasi-periodic system (1.3) is reducible.

Recently, a similar problem was considered by Her and You [7]. Let $C^\omega(\Lambda, gl(m,C))$ be the set of $m\times m$ matrices $A(\lambda)$ depending analytically on a parameter λ in a closed interval $\Lambda\subset R$. In [7], Her and You considered one-parameter families of quasi-periodic linear equations

$$\dot{x}=\big(A(\lambda)+g(\omega_1 t,\ldots,\omega_l t,\lambda)\big)x, \tag{1.4}$$

where $A\in C^\omega(\Lambda, gl(m,C))$, and g is analytic and sufficiently small. They proved that under some non-resonance conditions and some non-degeneracy conditions, there exists an open and dense set $\mathcal{A}$ in $C^\omega(\Lambda, gl(m,C))$, such that for each $A\in\mathcal{A}$, the system (1.4) is reducible for almost all $\lambda\in\Lambda$.

However, these papers only deal with the reducibility of the analytic system. With respect to the Hamiltonian system, Moser [12] proved the existence of the maximal-dimensional invariant tori under the assumption that the perturbation $h\in C^l$ with $l>2n+2$ (n denotes the dimension of angular variables). In [13], Pöschel obtained the maximal-dimensional invariant tori for the Hamiltonian system with $l>2n$. In [4], Chierchia and Qian obtained for Hamiltonian functions of class C^l with any $l>6n+5$, the lower-dimensional quasi-periodic solutions are proved to be of class C^p for any p with $2<p<p_*$ for a suitable $p_*=p_*(n,l)>2$ (which tends to infinity when $l\to\infty$). In [1], J. Albrecht proved the existence of the maximal-dimensional invariant tori in the Hamiltonian system, which are analytic and integrable except a $2n$ times continuously differentiable perturbation, provided that the moduli of continuity of the $2n$-th partial derivatives of the perturbation satisfy a condition of finiteness (condition on an integral), which is more general than a Hölder condition. In [3], Cheng obtained the non-existence of the maximal-dimensional invariant tori for the Hamiltonian system under C^l perturbations with $0<l<2n$. In contrast to this result, KAM theory guarantees (under more general assumptions) the persistence of the maximal-dimensional invariant tori for the Hamiltonian system under C^l perturbations with $l>2n$.

Motivated by [1], in this paper, we consider the reducibility for the finitely differentiable quasi-periodic linear system

$$\dot{x}=\big(A+\varepsilon\widetilde{Q}(t)\big)x,\quad x\in R^n, \tag{1.5}$$

where A is a constant $n\times n$ matrix with different eigenvalues, $\widetilde{Q}(t)$ is an $n\times n$ quasi-periodic matrix with respect to t, and $\varepsilon\in(0,\varepsilon_0)$ is a small perturbation parameter.