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A generalized Hilbert matrix acting on Hardy spaces ☆

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ABSTRACT

If μ is a positive Borel measure on the interval $[0,1)$, the Hankel matrix $\mathcal{H}_\mu=(\mu_{n,k})_{n,k\geqslant 0}$ with entries $\mu_{n,k}=\int_{[0,1)}t^{n+k}\,d\mu(t)$ induces formally the operator

$$\mathcal{H}_\mu(f)(z)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\mu_{n,k}a_k\right)z^n$$

on the space of all analytic functions $f(z)=\sum_{k=0}^{\infty}a_kz^k$, in the unit disc $\mathbb{D}$. In this paper we describe those measures μ for which $\mathcal{H}_\mu$ is a bounded (compact) operator from H^p into H^q, $0<p,q<\infty$. We also characterize the measures μ for which $\mathcal{H}_\mu$ lies in the Schatten class $S_p(H^2)$, $1<p<\infty$.

1. Introduction and main results

Let $\mathbb{D}=\{z\in\mathbb{C}\colon |z|<1\}$ denote the open unit disc in the complex plane $\mathbb{C}$ and let $\mathcal{H}ol(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$. We also let H^p $(0<p\leqslant\infty)$ be the classical Hardy spaces (see [9]).

If μ is a finite positive Borel measure on $[0,1)$ and $n=0,1,2,\ldots$, we let μ_n denote the moment of order n of μ, that is,

$$\mu_n=\int_{[0,1)}t^n\,d\mu(t),$$

and we define $\mathcal{H}_\mu$ to be the Hankel matrix $(\mu_{n,k})_{n,k\geqslant 0}$ with entries $\mu_{n,k}=\mu_{n+k}$. The matrix $\mathcal{H}_\mu$ can be viewed as an operator on spaces of analytic functions by its action on the Taylor coefficients: $a_n\mapsto\sum_{k=0}^{\infty}\mu_{n,k}a_k$, $n=0,1,2,\ldots$. To be precise, if $f(z)=\sum_{k=0}^{\infty}a_kz^k\in\mathcal{H}ol(\mathbb{D})$ we define

$$\mathcal{H}_\mu(f)(z)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\mu_{n,k}a_k\right)z^n, \tag{1.1}$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$.

If μ is the Lebesgue measure on $[0,1)$ the matrix $\mathcal{H}_\mu$ reduces to the classical Hilbert matrix $\mathcal{H}=((n+k+1)^{-1})_{n,k\geqslant 0}$, which induces the classical Hilbert operator $\mathcal{H}$, a prototype of a Hankel operator which has attracted a considerable amount

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of attention during the last years. Indeed, the study of the boundedness, the operator norm and the spectrum of $\mathcal{H}$ on Hardy and weighted Bergman spaces [1,6,7,14,23] links $\mathcal{H}$ up to weighted composition operators, the Szegö projection, Legendre functions and the theory of Muckenhoupt weights.

Hardy's inequality [9, p. 48] guarantees that $\mathcal{H}(f)$ is a well defined analytic function in $\mathbb{D}$ for every $f \in H^1$. However, the resulting Hilbert operator $\mathcal{H}$ is bounded from H^p to H^p if and only if $1 < p < \infty$ [6]. In a recent paper [17] Lanucha, Nowak, and Pavlovic have considered the question of finding subspaces of H^1 which are mapped by $\mathcal{H}$ into H^1.

Galanopoulos and Peláez [13] have described the measures μ so that the generalized Hilbert operator $\mathcal{H}_\mu$ becomes well defined and bounded on H^1. Carleson measures play a basic role in the work.

If $I \subset \partial\mathbb{D}$ is an interval, $|I|$ will denote the length of I. The *Carleson square* $S(I)$ is defined as $S(I) = \{re^{it}\colon\ e^{it} \in I,\ 1 - \frac{|I|}{2\pi} \leqslant r < 1\}$. Also, for $a \in \mathbb{D}$, the Carleson box $S(a)$ is defined by

$$S(a) = \left\{z \in \mathbb{D}\colon\ 1 - |z| \leqslant 1 - |a|,\ \left|\frac{\arg(a\bar{z})}{2\pi}\right| \leqslant \frac{1 - |a|}{2}\right\}.$$

If $s > 0$ and μ is a positive Borel measure on $\mathbb{D}$, we shall say that μ is an s-Carleson measure if there exists a positive constant C such that

$$\mu\big(S(I)\big) \leqslant C|I|^s, \quad \text{for any interval } I \subset \partial\mathbb{D},$$

or, equivalently, if there exists $C > 0$ such that

$$\mu\big(S(a)\big) \leqslant C\big(1 - |a|\big)^s, \quad \text{for all } a \in \mathbb{D}.$$

If μ satisfies $\lim_{|I|\to 0} \frac{\mu(S(I))}{|I|^s} = 0$ or, equivalently, $\lim_{|a|\to 1} \frac{\mu(S(a))}{(1-|a|^2)^s} = 0$, then we say that μ is a *vanishing s-Carleson measure*.

A 1-Carleson measure, respectively, a vanishing 1-Carleson measure, will be simply called a Carleson measure, respectively, a vanishing Carleson measure.

As an important ingredient in his work on interpolation by bounded analytic functions, Carleson [5] (see also Theorem 9.3 of [9]) proved that if $0 < p < \infty$ and μ is a positive Borel measure in $\mathbb{D}$ then $H^p \subset L^p(d\mu)$ if and only if μ is a Carleson measure. This result was extended by Duren [8] (see also [9, Theorem 9.4]) who proved that for $0 < p \leqslant q < \infty$, $H^p \subset L^q(d\mu)$ if and only if μ is a q/p-Carleson measure.

If X is a subspace of $\mathcal{H}ol(\mathbb{D})$, $0 < q < \infty$, and μ is a positive Borel measure in $\mathbb{D}$, μ is said to be a "*q-Carleson measure for the space X*" or an "*(X, q)-Carleson measure*" if $X \subset L^q(d\mu)$. The q-Carleson measures for the spaces H^p, $0 < p, q < \infty$, are completely characterized. The mentioned results of Carleson and Duren can be stated saying that if $0 < p \leqslant q < \infty$ then a positive Borel measure μ in $\mathbb{D}$ is a q-Carleson measure for H^p if and only if μ is a q/p-Carleson measure. Luecking [19] and Videnskii [26] solved the remaining case $0 < q < p$. We mention [4] for a complete information on Carleson measures for Hardy spaces.

Galanopoulos and Peláez proved in [13] that if μ is a Carleson measure then the operator $\mathcal{H}_\mu$ is well defined in H^1, obtaining en route the following integral representation

$$\mathcal{H}_\mu(f)(z) = \int_{[0,1)} \frac{f(t)}{1 - tz}\, d\mu(t), \quad z \in \mathbb{D},\ \text{for all } f \in H^1. \tag{1.2}$$

For simplicity, we shall write throughout the paper

$$I_\mu(f)(z) = \int_{[0,1)} \frac{f(t)}{1 - tz}\, d\mu(t), \tag{1.3}$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$. It was also proved in [13] that if $I_\mu(f)$ defines an analytic function in $\mathbb{D}$ for all $f \in H^1$, then μ has to be a Carleson measure. This condition does not ensure the boundedness of $\mathcal{H}_\mu$ on H^1, as the classical Hilbert operator $\mathcal{H}$ shows.

Let μ be a positive Borel measure in $\mathbb{D}$, $0 \leqslant \alpha < \infty$, and $0 < s < \infty$. Following [27], we say that μ is an α-logarithmic s-Carleson measure, respectively, a vanishing α-logarithmic s-Carleson measure, if

$$\sup_{a\in\mathbb{D}} \frac{\mu(S(a))\big(\log\frac{2}{1-|a|^2}\big)^\alpha}{(1 - |a|^2)^s} < \infty, \quad \text{respectively,} \quad \lim_{|a|\to 1^-} \frac{\mu(S(a))\big(\log\frac{2}{1-|a|^2}\big)^\alpha}{(1 - |a|^2)^s} = 0.$$

Theorem 1. 2 of [13] asserts that if μ is a Carleson measure on $[0, 1)$, then $\mathcal{H}_\mu$ is a bounded (respectively, compact) operator from H^1 into H^1 if and only if μ is a 1-logarithmic 1-Carleson measure (respectively, a vanishing 1-logarithmic 1-Carleson measure).

It is also known that $\mathcal{H}_\mu$ is bounded from H^2 into itself if and only if μ is a Carleson measure (see [24, p. 42, Theorem 7.2]).

Our main aim in this paper is to study the generalized Hilbert matrix $\mathcal{H}_\mu$ acting on H^p spaces ($0 < p < \infty$). Namely, for any given p, q with $0 < p, q < \infty$, we wish to characterize those for which $\mathcal{H}_\mu$ is a bounded (compact) operator from H^p