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Conditionally evenly convex sets and evenly quasi-convex maps

Marco Frittelli, Marco Maggis *

Dipartimento di Matematica, Università degli Studi di Milano, Italy

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ABSTRACT

Evenly convex sets in a topological vector space are defined as the intersection of a family of open half spaces. We introduce a generalization of this concept in the conditional framework and provide a generalized version of the bipolar theorem. This notion is then applied to obtain the dual representation of conditionally evenly quasi-convex maps, which turns out to be a fundamental ingredient in the study of quasi-convex dynamic risk measures.

1. Introduction

A subset C of a topological vector space is *evenly convex* if it is the intersection of a family of open half spaces, or equivalently, if every $x \notin C$ can be openly separated from C by a continuous linear functional. Obviously an evenly convex set is necessarily convex. This idea was firstly introduced by Fenchel [8] aimed to determine the largest family of convex sets C for which the polarity $C = C^{\circ\circ}$ holds true. More recent studies in this area led to a detailed analysis of evenly convex sets and evenly convex functions for the application in quasi-convex programming. Contributions to this branch of recent literature can be found in Daniilidis and Martínez-Legaz [5], Klee et al. [17], Martínez-Legaz and Vicente-Pérez [18] and Rodríguez and Vicente-Pérez [21].

It is well known that in the framework of incomplete financial markets the bipolar theorem is a key ingredient when we represent the super replication price of a contingent claim in terms of the class of martingale measures. Recently evenly convex sets and in particular evenly quasi-concave real valued functions have been considered by Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio in the context of decision theory [3] and risk measures [4]. Evenly quasi-concavity is the weakest notion that enables, in the static setting, a *complete* quasi-convex duality: the idea is to prove a *one to one* relationship between quasi-convex monotone functionals ρ and the function R in the dual representation. Obviously R will be unique only in an opportune class of maps satisfying certain properties. In decision theory the function R can be interpreted as the decision maker's index of uncertainty aversion: the uniqueness of R becomes crucial (see [3] and [6]) if we want to guarantee a robust dual representation of ρ characterized in terms of the unique R. The results in the present paper are meant to determine the mathematical background to deduce a dynamic version of this complete duality and are applied in [14].

In a conditional framework, as for example when $\mathcal{F}$ is a sigma algebra containing the sigma algebra $\mathcal{G}$ and we deal with $\mathcal{G}$-conditional expectation, $\mathcal{G}$-conditional sublinear expectation, $\mathcal{G}$-conditional risk measure, the analysis of the duality

* Corresponding author.
E-mail addresses: marco.frittelli@unimi.it (M. Frittelli), marco.maggis@unimi.it (M. Maggis).

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theory is more delicate. We may consider conditional maps $\rho : E \to L^0(\Omega, \mathcal{G}, \mathbb{P})$ defined either on vector spaces (i.e. $E = L^p(\Omega, \mathcal{F}, \mathbb{P})$) or on L^0-modules (i.e. $E = L^p_{\mathcal{G}}(\mathcal{F}) := \{yx \mid y \in L^0(\Omega, \mathcal{G}, \mathbb{P}) \text{ and } x \in L^p(\Omega, \mathcal{F}, \mathbb{P})\}$).

As described in detail by Filipovic, Kupper and Vogelpoth [9,10] and by Guo [16] the L^0-modules approach (see also Section 3 for more details) is a very powerful tool for the analysis of conditional maps and their dual representation.

In this paper we show that in order to achieve a conditional version of the representation of evenly quasi-convex maps a good notion of evenly convexity is crucial. We introduce the concept of a *conditionally evenly convex set*, which is tailor made for the conditional setting, in a framework that exceeds the module setting alone, so that will be applicable in many different context. We emphasize that, differently from the static case where the main tool is functional analysis, in the conditional setting this study involves substantial techniques from conditional probability.

In Section 2 we provide the characterization of evenly convexity (Theorem 1 and Proposition 9) and state the conditional version of the bipolar theorem (Theorem 2). Under additional topological assumptions, we show that conditionally convex sets that are closed or open are conditionally evenly convex (see Section 4, Proposition 4). As a consequence, the conditional evenly quasi-convexity of a function, i.e. the property that the conditional lower level sets are evenly convex, is a weaker assumption than quasi-convexity and lower (or upper) semicontinuity.

In Section 3 we apply the notion of conditionally evenly convex set to the *dual representation of evenly quasi-convex maps*, i.e. conditional maps $\rho : E \to L^0(\Omega, \mathcal{G}, \mathbb{P})$ with the property that the conditional lower level sets are evenly convex. Let $\bar{L}^0(\mathcal{G})$ be the space of extended random variables which may take values in $\mathbb{R} \cup \{\infty\}$. We prove in Theorem 3 that an evenly quasi-convex regular map $\pi : E \to \bar{L}^0(\mathcal{G})$ can be represented as

$$\pi(X) = \sup_{\mu \in \mathcal{L}(E, L^0(\mathcal{G}))} \mathcal{R}\big(\mu(X), \mu\big), \tag{1}$$

where

$$\mathcal{R}(Y, \mu) := \inf_{\xi \in E} \big\{\pi(\xi) \;\big|\; \mu(\xi) \geqslant Y\big\}, \quad Y \in L^0(\mathcal{G}),$$

E is a topological L^0-module and $\mathcal{L}(E, L^0(\mathcal{G}))$ is the module of continuous L^0-linear functionals over E.

The proof of this result is based on a version of the hyperplane separation theorem and not on some approximation or scalarization arguments, as it happened in the vector space setting (see [13]). By carefully analyzing the proof one may appreciate many similarities with the original demonstration in the static setting by Penot and Volle [19]. One key difference with [19], in addition to the conditional setting, is the continuity assumption needed to obtain the representation (1). We work, as in [3], with evenly quasi-convex functions, an assumption weaker than quasi-convexity and lower (or upper) semicontinuity.

1.1. Dynamic risk measures and the L^0-module approach

As explained in [13] the representation of the type (1) is a cornerstone in order to reach a robust representation of quasi-convex risk measures or acceptability indexes.

At the end of the Nineties in the seminal paper by Artzner, Delbaen, Eber and Heath [1], a rigorous axiomatic formalization of coherent risk measures was developed with a normative intent. The regulating agencies asked for computational methods to estimate the capital requirements, exceeding the unmistakable lacks showed by the extremely popular $V@R$. Risk measures are real valued functionals ρ defined on a space of random variables which encloses every possible financial position. The risk of a financial position was originally defined in [1] as the minimal amount of money that an institution will have to sum up to a position X in order to make it acceptable with respect to some *criterium* modeled by an acceptance set $\mathcal{A}$.

The class of coherent risk measures was later extended to the class of convex risk measures, independently introduced by Föllmer and Schied (2002, [11]) and Frittelli and Rosazza Gianin (2002, [15]). Since then, the interest on this subject enormously expanded and the vast literature can be found in [12] 3rd edition, as well as in Ruszczyinski and Shapiro [22], Pflug [20], Bot, Lorenz and Wanka [2].

One key axiom in the class of convex risk measures – the cash additivity property – was relaxed by El Karoui and Ravanelli (2009, [7]) in markets with stochastic discount factors; finally Cerreia-Vioglio et al. (2010, [4]) showed that quasi-convexity describes better than convexity the principle of diversification, whenever cash additivity does not hold true. Following this trajectory we may conclude that the largest class of feasible risk measure is the following.

Definition 1. Let E be any vector space of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the $\mathbb{P}$ almost sure partial order. A quasi-convex risk measure is a functional $\rho : E \to \mathbb{R}$ which satisfies

(i) *monotonicity*, i.e. $X_1 \leqslant X_2$ implies $\rho(X_1) \geqslant \rho(X_2)$ for every $X_1, X_2 \in E$,
(ii) *quasi-convexity*, i.e. $\rho(tX_1 + (1-t)X_2) \leqslant \max\{\rho(X_1), \rho(X_2)\}$ for all $t \in [0,1]$.

In the dynamic description of risk, we have the following situation: let $0 \leqslant t \leqslant T$ and fix a non-empty convex set $C_T \in E \subset L^0(\mathcal{F})$ such that $C_T + L^0_+ \subseteq C_T$. The set C_T represents the future positions considered acceptable by the supervising