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Decay rate and radial symmetry of the exponential elliptic equation

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ABSTRACT

Let $n \geqslant 3$, $\alpha, \beta \in \mathbb{R}$, and let v be a solution of $\Delta v + \alpha e^v + \beta x \cdot \nabla e^v = 0$ in $\mathbb{R}^n$, which satisfies the conditions $\lim_{R\to\infty} \frac{1}{\log R} \int_1^R \rho^{1-n}(\int_{B_\rho} e^v\,dx)\,d\rho \in (0,\infty)$ and $|x|^2 e^{v(x)} \leqslant A_1$ in $\mathbb{R}^n$. We prove that $\frac{v(x)}{\log|x|} \to -2$ as $|x| \to \infty$ and $\alpha > 2\beta$. As a consequence we prove that there exists a constant $R_0 > 0$ such that if the solution $v(x)$ is radially symmetric for $|x| < R_0$ and satisfies some gradient bound, then v is radially symmetric about the origin in $\mathbb{R}^n$.

1. Introduction

In this paper we will study various properties of the solution v of the nonlinear elliptic equation

$$\Delta v + \alpha e^v + \beta x \cdot \nabla e^v = 0 \quad \text{in } \mathbb{R}^n \tag{1.1}$$

for any $n \geqslant 3$ where $\alpha, \beta \in \mathbb{R}$ are some constants. Let $v = \log u$. Then u satisfies

$$\Delta \log u + \alpha u + \beta x \cdot \nabla u = 0, \quad u > 0, \text{ in } \mathbb{R}^n. \tag{1.2}$$

As observed by S.Y. Hsu [9], the radial symmetric solution of (1.2) is the singular limit of the radial symmetric solutions of the nonlinear elliptic equation,

$$\Delta(u^m/m) + \alpha u + \beta x \cdot \nabla u = 0, \quad u > 0, \text{ in } \mathbb{R}^n, \tag{1.3}$$

as $m \searrow 0$. On the other hand as observed by P. Daskalopoulos and N. Sesum [4], K.M. Hui and S.H. Kim [10,11], (1.2) also arises in the study of the extinction behaviour and global behaviour of the solutions of the logarithmic diffusion equation,

$$u_t = \Delta \log u, \quad u > 0, \text{ in } \mathbb{R}^n. \tag{1.4}$$

(1.2) also arises in the study of self-similar solutions of (1.4) [4,10,11,15,16]. Hence in order to understand the behaviour of the solutions of (1.3) and (1.4) it is important to understand the properties of solutions of (1.1).

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In [8] S.Y. Hsu proved that there exists a radially symmetric solution of (1.1) (or equivalently (1.2)) if and only if either $\alpha \geqslant 0$ or $\beta > 0$. She also proved that when $n \geqslant 3$ and $\alpha > \max(2\beta, 0)$, then any radially symmetric solution v of (1.1) satisfies

$$\lim_{|x|\to\infty} |x|^2 e^{v(x)} = \frac{2(n-2)}{\alpha - 2\beta}. \tag{1.5}$$

By (1.5) and a direct computation the radially symmetric solution v of (1.1) satisfies

$$\lim_{|x|\to\infty} \frac{v(x)}{\log|x|} = -2, \tag{1.6}$$

$$A_0 := \lim_{R\to\infty} \frac{1}{\log R} \int_1^R \frac{1}{\rho^{n-1}} \left(\int_{|x|<\rho} e^v \, dx \right) d\rho \in (0, \infty) \tag{1.7}$$

and

$$|x|^2 e^{v(x)} \leqslant A_1 \quad \forall x \in \mathbb{R}^n \tag{1.8}$$

for some constant $A_1 > 0$. A natural question is if v is a solution of (1.1) which satisfies (1.7) and (1.8) for some constant $A_1 > 0$, will v satisfy (1.6) and is v radially symmetric about the origin? We answer the first question affirmatively in this paper. For the second question we prove that there exists a constant $R_0 > 0$ such that if the solution $v(x)$ is radially symmetric for $|x| < R_0$ and satisfies some gradient bound, then v is radially symmetric about the origin in $\mathbb{R}^n$.

We say that v is a solution of (1.1) if v is continuous in $\mathbb{R}^n$ and satisfies

$$\int_{\mathbb{R}^n} \left[v\Delta\eta + (\alpha - n\beta)e^v - \beta(x\cdot\nabla)e^v \right] dx = 0 \quad \forall \eta \in C_0^\infty(\mathbb{R}^n).$$

Note that if v is a solution of (1.1), then by standard elliptic regularity theory [6] and a bootstrap argument $v \in C^\infty(\mathbb{R}^n)$. For any solution v of (1.1) we define the rotation operator Φ_{ij} by

$$\Phi_{ij}(x) = x_i v_{x_j}(x) - x_j v_{x_i}(x) \quad \forall x = (x_1, \dots, x_n) \in \mathbb{R}^n, \ i \neq j, \ i, j = 1, \dots, n.$$

Note that if we write $x_1 = \rho\cos\theta$ and $x_2 = \rho\sin\theta$ where $\rho = \sqrt{x_1^2 + x_2^2}$, then $\Phi_{12}(x) = \frac{\partial v}{\partial\theta}(x)$.

We are now ready to state the main results of this paper.

Theorem 1.1. *Let $n \geqslant 3$ and $\alpha, \beta \in \mathbb{R}$. Suppose v is a solution of (1.1) which satisfies (1.7) and (1.8) for some constant $A_1 > 0$. Then v satisfies (1.6) and $\alpha > 2\beta$.*

Corollary 1.2. *Let $n \geqslant 3$. Suppose $\alpha \leqslant 2\beta$. Then (1.1) does not have any solution that satisfies both (1.7) and (1.8) for some constant $A_1 > 0$.*

Proposition 1.3. *Let $n \geqslant 3$ and $2\beta < \alpha < n\beta$. Suppose v is a solution of (1.1) which satisfies (1.7), (1.8),*

$$\|x \cdot \nabla v\|_{L^\infty(\mathbb{R}^n)} \leqslant C < \infty, \tag{1.9}$$

and

$$\lim_{|x|\to\infty} |x|^{n-2} \left|\Phi_{ij}(x)\right| = 0 \quad \forall i \neq j, \ i, j = 1, \dots, n. \tag{1.10}$$

Then there exists a constant $R_0 > 0$ such that if v is radially symmetric in B_{R_0}, then v is radially symmetric in $\mathbb{R}^n$.

Note that although there are many research done on the radial symmetry of elliptic equations without first order term by B. Gidas, W.M. Ni, and L. Nirenberg [5], L. Caffaralli, B. Gidas and J. Spruck [2], W. Chen and C. Li [3], S.D. Taliaferro [14] and others, very little is known about the radial symmetry of elliptic equations with non-zero first order term. The reason is that one cannot use the moving plane technique to prove the radial symmetry of the solution for elliptic equations with non-zero first order term. We refer the readers to the paper [1] by J.S. Baek for various methods on proving symmetry of solutions of elliptic equations.

The recent paper [13] by E. Kamalinejad and A. Moradifam is one of the few papers that studies the radial symmetry of elliptic equations with non-zero first order term. Hence our result on radial symmetry is new.

The plan of the paper is as follows. In Section 2 we will prove Theorem 1.1 and Corollary 1.2. In Section 3 we will prove Proposition 1.3. For any $r > 0$, $x_0 \in \mathbb{R}^n$, let $B_r(x_0) = \{x \in \mathbb{R}^n: |x - x_0| < r\}$ and $B_r = B_r(0)$. Let $S^{n-1} = \{x \in \mathbb{R}^n: |x| = 1\}$. We will let $n \geqslant 3$, $\alpha, \beta \in \mathbb{R}$, and let v be a solution of (1.1) which satisfies both (1.7) and (1.8) for some constant $A_1 > 0$ for the rest of the paper. We will also let A_0 be given by (1.7) for the rest of the paper.