J. Math. Anal. Appl. 413 (2014) 321–342

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Bifurcation diagram and stability for a one-parameter family of planar vector fields

J.D. García-Saldaña [a,*], A. Gasull [a], H. Giacomini [b]

[a] Departament de Matemàtiques, Universitat Autònoma de Barcelona, Edifici C, 08193 Bellaterra, Barcelona, Spain
[b] Laboratoire de Mathématiques et Physique Théorique, Faculté des Sciences et Techniques, Université de Tours, C.N.R.S. UMR 7350, 37200 Tours, France

ARTICLE INFO

Article history:
Received 1 July 2013
Available online 27 November 2013
Submitted by R. Popovych

Keywords:
Planar polynomial system
Uniqueness and hyperbolicity of the limit cycle
Polycycle
Bifurcation
Phase portrait on the Poincaré disc
Dulac function
Stability
Nilpotent point
Basin of attraction

ABSTRACT

We consider the one-parameter family of planar quintic systems, $\dot{x} = y^3 - x^3$, $\dot{y} = -x + my^5$, introduced by A. Bacciotti in 1985. It is known that it has at most one limit cycle and that it can exist only when the parameter m is in $(0.36, 0.6)$. In this paper, using the Bendixson–Dulac theorem, we give a new unified proof of all the previous results. We shrink this interval to $(0.547, 0.6)$ and we prove the hyperbolicity of the limit cycle. Furthermore, we consider the question of the existence of polycycles. The main interest and difficulty for studying this family is that it is not a semi-complete family of rotated vector fields. When the system has a limit cycle, we also determine explicit lower bounds of the basin of attraction of the origin. Finally, we answer an open question about the change of stability of the origin for an extension of the above systems.

1. Introduction and main results

A. Bacciotti, during a conference about the stability of analytic dynamical systems held in Florence in 1985, proposed to study the stability of the origin of the following quintic system

$$\begin{cases} \dot{x} = y^3 - x^3, \\ \dot{y} = -x + my^5, \quad m \in \mathbb{R}. \end{cases} \tag{1}$$

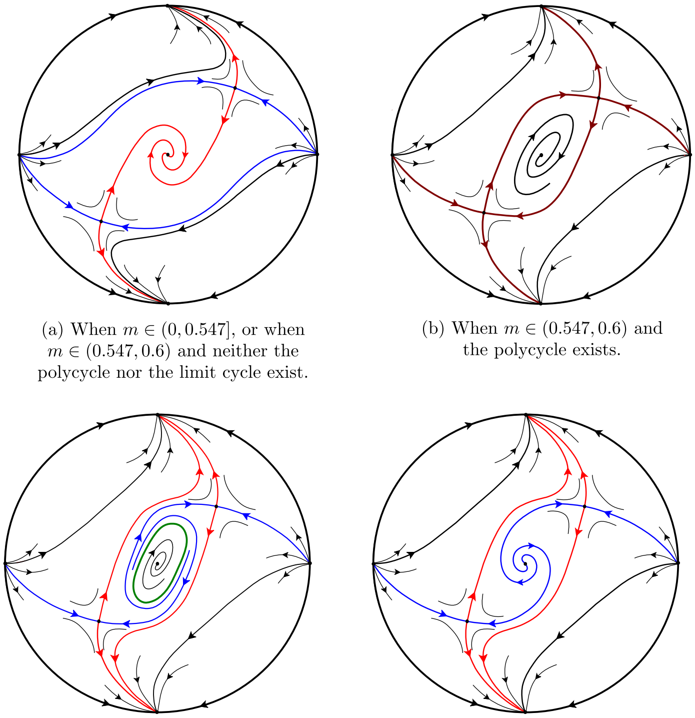
Two years later, Galeotti and Gori in [10] published an extensive study of (1). They proved that system (1) has no limit cycles when $m \in (-\infty, 0.36] \cup [0.6, \infty)$, otherwise, it has at most one. Their proofs are mainly based on the study of the stability of the limit cycles which is controlled by the sign of its characteristic exponent, together with a transformation of the system using a special type of adapted polar coordinates. Their proof of the uniqueness of the limit cycle does not cover its hyperbolicity.

In this paper we refine the above results. To guess which is the actual bifurcation diagram we first did a numerical study, obtaining the following results. It seems that there exists a value $m^* > 0$ such that:

(i) System (1) has no limit cycles if $m \in (-\infty, m^*] \cup [0.6, \infty)$. Moreover, for $m = m^*$ it has a heteroclinic polycycle formed by the separatrices of the two saddle points located at $(\pm m^{-1/4}, \pm m^{-1/4})$.

* Corresponding author.
E-mail addresses: johanna@mat.uab.cat (J.D. García-Saldaña), gasull@mat.uab.cat (A. Gasull), Hector.Giacomini@lmpt.univ-tours.fr (H. Giacomini).

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.11.047



(a) When $m \in (0, 0.547]$, or when $m \in (0.547, 0.6)$ and neither the polycycle nor the limit cycle exist.

(b) When $m \in (0.547, 0.6)$ and the polycycle exists.

(c) When $m \in (0.547, 0.6)$ and the limit cycle exists.

(d) For $m \in [0.6, \infty)$.

Fig. 1. Phase portraits of system (1).

(ii) For $m \in (m^*, 0.6)$ the system has exactly one unstable limit cycle.
(iii) The value m^* is approximately 0.560115.

Recall that a polycycle is a simple, closed curve, formed by several solutions of the system, which admits a Poincaré return map. The claims (i) and (ii) above coincide with the results described in [10]. Concerning the location of the value m^* however, our computations differ from the results proposed in [10] where it is claimed that m^* is between 0.58 and 0.59.

The first aim of this work is to obtain analytic results that confirm, as accurate as possible, the above claims. To clarify the phase portraits of the system, we will study them on the Poincaré disc, see [3,24].

For $m \leqslant 0$, system (1) has no periodic orbits because $x^2/2 + y^4/4$ is a global Lyapunov function. Therefore, the origin is a global attractor. In particular, its phase portrait is trivial. Therefore, we will concentrate on the case $m > 0$. In this case, the system has three critical points, $(\pm m^{-1/4}, \pm m^{-1/4})$ and $(0, 0)$. The first two points are saddles and the third one is a monodromic nilpotent singularity. Its stability can be determined using the tools introduced in [2,19], see Section 2 and Theorem 1.3 below. We prove:

Theorem 1.1. *Consider system (1).*

(i) *It has neither periodic orbits, nor polycycles, when $m \in (-\infty, 0.547] \cup [0.6, \infty)$. Otherwise, it has at most one periodic orbit or one polycycle, but cannot coexist. Moreover, when the limit cycle exists, it is hyperbolic and unstable.*
(ii) *For $m > 0$, its phase portraits on the Poincaré disc, are given in Fig. 1.*
(iii) *Let $\mathcal{M}$ be the set of values of m for which it has a heteroclinic polycycle. Then $\mathcal{M}$ is finite, non-empty and it is contained in $(0.547, 0.6)$. Moreover, the system corresponding to $m \in \mathcal{M}$ has no limit cycles and its phase portrait is given by Fig. 1 (b).*

Our simulations show that (a), (b) and (c) of Fig. 1 occur when $m \in (0, m^*)$, $m = m^*$ and $m > m^*$, respectively, for some $m^* \in (0.547, 0.6)$, that numerically we have found to be $m^* \approx 0.560115$. We have not been able to prove the existence of this special value m^* rigorously, because our system is not a semi-complete family of rotated vector fields (SCFRVF) and this fact hinders the obtention of the full bifurcation diagram; see the discussion in Subsection 3.1 and Example 7.1. This is precisely the reason why we have decided to push forward the study of system (1). Our approach can be useful to understand other interesting polynomial systems of differential equations that have been considered previously; see for instance [4,8].