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On the dynamics of a family of generated renormalization transformations ☆

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ABSTRACT

We study the family of renormalization transformations of the generalized d-dimensional diamond hierarchical Potts model in statistical mechanic and prove that their Julia sets and non-escaping loci are always connected, where $d \geqslant 2$. In particular, we prove that their Julia sets can never be a Sierpiński carpet if the parameter is real. We show that the Julia set is a quasicircle if and only if the parameter lies in the unbounded capture domain of these models. Moreover, the asymptotic formula of the Hausdorff dimension of the Julia set is calculated as the parameter tends to infinity.

1. Introduction

The statistical mechanical models on hierarchical lattices have attracted many interests recently since they exhibit a deep connection between their limiting sets of the zeros of the partition functions and the Julia sets of rational maps in complex dynamics [2–4,7,23–25]. A celebrated Lee–Yang theorem [15,38] in statistical mechanics asserts that the zeros of the partition function for some magnetic materials lie on the unit circle in the complex plane, which is corresponding to a purely imaginary magnetic field. This means that the complex singularities of the free energy lie on this line, where the free energy is the logarithm of the partition function.

The partition function $\mathrm{Z} = \mathrm{Z}(z, \mathrm{t})$ can be written as a Laurent polynomial in two variables z and t, where z is a 'field-like' variable and t is 'temperature-like'. Note that the complex zeros of $\mathrm{Z}(z, \mathrm{t})$ in z are called the *Lee–Yang zeros* for a fixed $\mathrm{t} \in [0, 1]$. Naturally, one can study the zeros of $\mathrm{Z}(z, \mathrm{t})$ in the t-variable. These zeros are called *Fisher zeros* since they were first studied by Fisher for regular two-dimensional lattice [12,5]. However, compared with the Lee–Yang zeros, Fisher zeros do not lie on the unit circle any more. For example, for the regular two-dimensional lattice, the Fisher zeros lie on the union of two circles $|\mathrm{t} \pm 1| = \sqrt{2}$. For more comprehensive introduction on Lee–Yang zeros and Fisher zeros, see [4] and the references therein.

In 1983, Derrida, de Seze and Itzykson showed that the Fisher circles of the Ising model on the regular two-dimensional lattice $\mathbb{Z}^2$ become a fractal Julia set upon replacing $\mathbb{Z}^2$ by a hierarchical lattice [7]. They proved that the corresponding singularities of the free energy lie on the Julia set of the rational map

$$z \mapsto \left(\frac{z^2 + \lambda - 1}{2z + \lambda - 2}\right)^2. \tag{1.1}$$

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This means that the distribution of the singularities of the free energy can have a pretty wild geometry. Henceforth, a lot of works related on the Julia sets of this renormalization transformation appeared (see [1,2,13,14,23–25,32] and references therein). For the ideas formulated in renormalization transformation in statistical mechanics, see [35].

Recently, Qiao considered the generalized diamond hierarchical Potts model and proved that the family of rational maps

$$U_{mn\lambda}(z) = \left(\frac{(z+\lambda-1)^m + (\lambda-1)(z-1)^m}{(z+\lambda-1)^m - (z-1)^m}\right)^n \tag{1.2}$$

is actually the renormalization transformation of the *generalized diamond hierarchical Potts model* [23, Theorem 1.1], where $m, n \geqslant 2$ are both integers and $\lambda \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ is a complex parameter. The standard diamond lattice ($m = n = 2$) and the *diamond-like lattice* ($m = 2$ and $n \in \mathbb{N}$) are the special cases of (1.2).

In this paper, we will consider the case for $d := m = n \geqslant 2$. For simplicity, we use $U_{d\lambda}$ to denote $U_{dd\lambda}$ in (1.2). We not only study the topological properties of the Julia sets of $U_{d\lambda}$, but also consider the connectivity of the non-escaping locus of the parameter space of this renormalization transformation.

If $\lambda = 0$, then $U_{d\lambda}$ degenerates to a parabolic polynomial $U_{d0}(z) = (\frac{z+d-1}{d})^d$ whose Julia set is a Jordan curve. For the connectivity of the Julia sets of $U_{d\lambda}$, we have the following theorem.

Theorem 1.1. *The Julia set of $U_{d\lambda}$ is always connected for every $d \geqslant 2$ and $\lambda \in \mathbb{C}^*$.*

Note that Qiao and Li proved that the Julia set of $U_{d\lambda}$ is connected for $d = 2$ and $\lambda \in \mathbb{R}$ [24]. We would like to remark that if $m \neq n$, then there exists parameter $\lambda \in \mathbb{C}^*$ such the Julia set of $U_{mn\lambda}$ defined in (1.2) is disconnected (see [23, Fig. 3.1] for example).

Let $\overline{\mathbb{C}} = \mathbb{C} \cup \{0\}$ be the Riemann sphere. According to [33], a connected and locally connected compact set S in $\overline{\mathbb{C}}$ is called a *Sierpiński carpet* if it has empty interior and can be written as $S = \overline{\mathbb{C}} \setminus \bigcup_{i\in\mathbb{N}} D_i$, where $\{D_i\}_{i\in\mathbb{N}}$ are Jordan regions satisfying $\partial D_i \cap \partial D_j = \emptyset$ for $i \neq j$ and the spherical diameter $\operatorname{diam}(\partial D_i) \to 0$ as $i \to \infty$.

The first example of the Sierpiński carpet as the Julia set of a rational map was given in [20, Appendix F]. Afterwards, many families of the rational maps serve the examples such that their Julia sets are Sierpiński carpets for suitable parameters. See [8] for the family of McMullen maps and [36] for generated McMullen maps. However, for the renormalization transformation $U_{d\lambda}$, we have the following theorem.

Theorem 1.2. *For $d \geqslant 2$ and $\lambda \in \mathbb{R}$, the Julia set of $U_{d\lambda}$ is not a Sierpiński carpet.*

The proof of Theorem 1.2 is based on proving the intersection of the boundaries of two of the Fatou components of $U_{d\lambda}$ are always non-empty (see Lemma 3.1 and Theorem 3.2).

The *Mandelbrot set* of quadratic polynomials $f_c(z) = z^2 + c$ is defined by

$$M = \left\{c \in \mathbb{C}\colon\ f_c^{\circ n}(0) \nrightarrow \infty \text{ as } n \to \infty\right\}.$$

Douady and Hubbard showed that M is connected [10]. For higher degree polynomials with only one critical point, there are associated *Multibrot sets*. For rational maps, one way to study the parameter space is to consider the *connectedness locus*, which consists of all parameters such the corresponding Julia set is connected. However, the connectedness locus makes no sense in our case since every Julia set is connected.

For $\lambda \neq 0$, then 1 and ∞ are two superattracting fixed points of $U_{d\lambda}$. The *non-escaping locus* $\mathcal{M}_d$ associated to this family is defined by

$$\mathcal{M}_d = \left\{\lambda \in \mathbb{C}^*\colon\ U_{d\lambda}^{\circ n}(0) \nrightarrow 1 \text{ and } U_{d\lambda}^{\circ n}(0) \nrightarrow \infty \text{ as } n \to \infty\right\} \cup \{0\}. \tag{1.3}$$

Obviously, “non-escaping” here means the collection of those parameters such that the orbit of 0 cannot be attracted by 1 and ∞. Note that 0 is a critical value of $U_{d\lambda}$.

The non-escaping locus $\mathcal{M}_d$ can be identified as the complex plane cutting out infinitely many simply connected domains, which will be called ‘capture domains’ later (see Fig. 1 and Proposition 4.4). There exist many small copies of the Mandelbrot set M in $\mathcal{M}_d$ which correspond to the renormalizable parameters.

For the connectivity of the non-escaping locus $\mathcal{M}_d$, Wang et al. proved that $\mathcal{M}_2$ is connected [32, Theorem 1.1]. We now generate this result to all $\mathcal{M}_d$, where $d \geqslant 2$.

Theorem 1.3. *The non-escaping locus $\mathcal{M}_d$ is connected for $d \geqslant 2$.*

The proof of the connectivity of $\mathcal{M}_2$ in [32] is based on constructing Riemann mapping from the capture domain to the unit disk $\mathbb{D}$, which is tediously long. Here, we give a proof of Theorem 1.3 by using the methods of Teichmüller theory of the rational maps which was developed in [19]. The proof is largely simplified and there are several additional results.