J. Math. Anal. Appl. 413 (2014) 378–391

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# Generic results in classes of ultradifferentiable functions

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ARTICLE INFO

*Article history:*
Received 23 October 2013
Available online 4 December 2013
Submitted by Richard M. Aron

*Keywords:*
Ultradifferentiable functions
Denjoy–Carleman classes
Beurling spaces
Roumieu spaces
Residual sets
Prevalent sets
Shy sets
Lineability

ABSTRACT

Let $E$ be a Denjoy–Carleman class of ultradifferentiable functions of Beurling type on the real line that strictly contains another class $F$ of Roumieu type. We show that the set $S$ of functions in $E$ that are nowhere in the class $F$ is large in the topological sense (it is residual), in the measure theoretic sense (it is prevalent), and that $S \cup \{0\}$ contains an infinite dimensional linear subspace (it is lineable). Consequences for the Gevrey classes are given. Similar results are also obtained for classes of ultradifferentiable functions defined imposing conditions on the Fourier–Laplace transform of the function.



## 1. Introduction

Let $E$ be a Denjoy–Carleman class of ultradifferentiable functions of Beurling type on the real line $\mathbb{R}$ that strictly contains another class $F$ of Roumieu type. The aim of this paper is to investigate how large is the set of functions in the class $E$ that are nowhere in the class $F$, i.e. such that the restriction of the function to any open subset of $\mathbb{R}$ does not belong to this class. In this way we complement work by Schmets and Valdivia [25], Bernal-González [5], Bastin, Nicolay and the author [3] and by Bastin, Conejero, Seoane-Sepúlveda and the author [4]. In order to be more precise, we need some definitions and notations.

Given an open subset $\Omega$ of $\mathbb{R}^n$, let $\mathcal{E}(\Omega)$ be the set of all complex-valued smooth functions on $\Omega$. If $K$ is a compact subset of $\mathbb{R}^n$, let $\mathcal{E}(K)$ denote the set of all complex-valued smooth functions on the interior of $K$ such that $D^\alpha f$ can be continuously extended to $K$ for all $\alpha \in \mathbb{N}_0^n$. Moreover, if $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, we use the notation $|\alpha| = \alpha_1 + \dots + \alpha_n$.

An arbitrary sequence of positive real numbers $M = (M_k)_{k\in\mathbb{N}_0}$ is called a *weight sequence*. For every weight sequence $M$, every compact subset $K$ of $\mathbb{R}^n$ and every $h > 0$, we define the space $\mathcal{E}_{M,h}(K)$ as the space of functions $f \in \mathcal{E}(K)$ such that

$$\|f\|_{K,h} := \sup_{\alpha\in\mathbb{N}_0^n} \sup_{x\in K} \frac{|D^\alpha f(x)|}{h^{|\alpha|} M_{|\alpha|}} < +\infty.$$

Endowed with the norm $\|\cdot\|_{K,h}$, the space $\mathcal{E}_{M,h}(K)$ is a Banach space.

**Definition 1.1.** If $\Omega$ is an open subset of $\mathbb{R}^n$, the space $\mathcal{E}_{\{M\}}(\Omega)$ is defined by

$$\mathcal{E}_{\{M\}}(\Omega) := \{ f \in \mathcal{E}(\Omega):\ \forall K \subset \Omega \text{ compact } \exists h > 0 \text{ such that } \|f\|_{K,h} < +\infty \}.$$

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0022-247X/$ – see front matter 
http://dx.doi.org/10.1016/j.jmaa.2013.11.067

If $f \in \mathcal{E}_{\{M\}}(\Omega)$, we say that $f$ is *M-ultradifferentiable of Roumieu type* on $\Omega$. We obtain a locally convex topology on these spaces via the representation

$$\mathcal{E}_{\{M\}}(\Omega) = \operatorname*{proj}_{\overleftarrow{K \subset \Omega}} \operatorname*{ind}_{\overrightarrow{h>0}} \mathcal{E}_{M,h}(K).$$

Fundamental examples of Roumieu spaces are given by the weight sequences $(k!)_{k\in\mathbb{N}_0}$ and $((k!)^\alpha)_{k\in\mathbb{N}_0}$ with $\alpha > 1$. They correspond respectively to the space of real analytic functions on $\Omega$ and the space of Gevrey differentiable functions of order $\alpha$ on $\Omega$.

On weight sequences, the following conditions are usually considered:

- A weight sequence $M$ is *logarithmically convex* (or shortly *log-convex*) if $M_k^2 \leqslant M_{k-1}M_{k+1}$ for every $k \in \mathbb{N}$. The Gorny theorem [14] states that for every weight sequence $M$, there is a log-convex weight sequence $L$ such that $\mathcal{E}_{\{M\}}(\Omega) = \mathcal{E}_{\{L\}}(\Omega)$. If the sequence $M$ is log-convex, then the sequence $(\frac{M_k}{M_{k-1}})_{k\in\mathbb{N}}$ is increasing and one has $M_kM_l \leqslant M_{k+l}$ for every $k, l \in \mathbb{N}_0$. This implies that the space $\mathcal{E}_{\{M\}}(\Omega)$ is an algebra.
- Since we have $\mathcal{E}_{\{M\}}(\Omega) = \mathcal{E}_{\{\frac{M}{M_0}\}}(\Omega)$, we can assume that any weight sequence $M$ is such that $M_0 = 1$.
- We say that the sequence $M$ is *quasianalytic* if it satisfies one of the two following equivalent conditions

$$1.\quad \sum_{n=1}^{+\infty} \frac{M_{n-1}}{M_n} = +\infty, \qquad 2.\quad \sum_{n=1}^{+\infty} (M_n)^{-1/n} = +\infty.$$

  If this is not the case, we say that the sequence is *non-quasianalytic*. The Denjoy–Carleman theorem states that if $M$ is log-convex, then the class $\mathcal{E}_{\{M\}}(\Omega)$ is quasianalytic (i.e. 0 is the unique function $f$ in the space for which there is a point $x \in \Omega$ such that $D^\alpha f(x) = 0$ for every $\alpha \in \mathbb{N}_0^n$) if and only if the sequence $M$ is quasianalytic (see for example [23, Theorem 19.11]). Note that the class $\mathcal{E}_{\{M\}}(\Omega)$ is quasianalytic if and only if there is no non-trivial function in $\mathcal{E}_{\{M\}}(\Omega)$ with compact support (a proof of this result can be found in [23, Theorem 19.10]). Then, if the class is non-quasianalytic, given an open subset $\Omega$ of $\mathbb{R}^n$ and a compact $K \subset \Omega$, there exists a function of $\mathcal{E}_{\{M\}}(\mathbb{R}^n)$ having a compact support included in $\Omega$ and being identically equal to 1 in $K$.

**Agreement.** In this paper, we will always assume that any weight sequence $M$ is log-convex and $M_0 = 1$.

Let us now introduce the second type of Denjoy–Carleman classes.

**Definition 1.2.** If $\Omega$ is an open subset of $\mathbb{R}^n$, we define the space $\mathcal{E}_{(M)}(\Omega)$ by

$$\mathcal{E}_{(M)}(\Omega) := \{ f \in \mathcal{E}(\Omega) \colon \forall K \subset \Omega \text{ compact}, \ \forall h > 0, \ \|f\|_{K,h} < +\infty \}.$$

If $f \in \mathcal{E}_{(M)}(\Omega)$, we say that $f$ is *M-ultradifferentiable of Beurling type* on $\Omega$ and we use the representation

$$\mathcal{E}_{(M)}(\Omega) = \operatorname*{proj}_{\overleftarrow{K \subset \Omega}} \operatorname*{proj}_{\overleftarrow{h>0}} \mathcal{E}_{M,h}(K)$$

to endow $\mathcal{E}_{(M)}(\Omega)$ with a structure of Fréchet space.

Of course, we always have $\mathcal{E}_{(M)}(\Omega) \subset \mathcal{E}_{\{M\}}(\Omega)$. Moreover, conditions on two weight sequences $M$ and $N$ to have the inclusion $\mathcal{E}_{\{M\}}(\Omega) \subset \mathcal{E}_{(N)}(\Omega)$ are known and presented in the second section of this paper. Let us consider the following definition.

**Definition 1.3.** We say that a function is *nowhere in* $\mathcal{E}_{\{M\}}$ if its restriction to any open and non-empty subset $\Omega$ of $\mathbb{R}$ never belongs to $\mathcal{E}_{\{M\}}(\Omega)$.

We want to handle the question of how large the subset of $\mathcal{E}_{(N)}(\mathbb{R})$ formed by the functions which are nowhere in $\mathcal{E}_{\{M\}}$ is. We will use three different notions of genericity. Let us recall their definitions here.

First, let us recall this classical definition of residuality from a topological point of view.

**Definition 1.4.** If $X$ is a Baire space, then a subset $A \subset X$ is called *residual* (or *comeager*) if $A$ contains a countable union of dense open sets of $X$, or equivalently if $X \setminus A$ is included in a countable union of closed sets of $X$ with empty interior.

From a measure-theoretical point of view, we will use the notion of prevalence. It was introduced by Christensen and rediscovered later by Hunt, Sauer and Yorke in order to generalize the notion of "Lebesgue almost everywhere" to infinite dimensional spaces. More precisely, we use the following definition.