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# A converse of Loewner–Heinz inequality and applications to operator means ☆

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ABSTRACT

Let $f(t)$ be an operator monotone function. Then $A \leqslant B$ implies $f(A) \leqslant f(B)$, but the converse implication is not true. Let $A \sharp B$ be the geometric mean of $A, B \geqslant 0$. If $A \leqslant B$, then $B^{-1} \sharp A \leqslant I$; the converse implication is not true either. We will show that if $f(\lambda B + I)^{-1} \sharp f(\lambda A + I) \leqslant I$ for all sufficiently small $\lambda > 0$, then $f(\lambda A + I) \leqslant f(\lambda B + I)$ and $A \leqslant B$. Moreover, we extend it to multi-variable matrices means.


## 1. Introduction

In what follows, $\mathcal{H}$ means a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and an operator means a bounded linear operator on $\mathcal{H}$. An operator $A$ is said to be positive (denoted by $A \geqslant 0$) if and only if $\langle Ax, x \rangle \geqslant 0$ for all $x \in \mathcal{H}$, and $A \leqslant B$ means $B - A$ is positive. Moreover, an operator $A$ is said to be positive definite (denoted by $A > 0$) if $A$ is positive and invertible.

A real continuous function $f(t)$ defined on a real interval $I$ is said to be *operator monotone*, provided $A \leqslant B$ implies $f(A) \leqslant f(B)$ for any two bounded self-adjoint operators $A$ and $B$ whose spectra are in $I$. The Loewner–Heinz inequality means the power function $t^a$ is operator monotone on $[0, \infty)$ for $0 < a < 1$. $\log t$ is operator monotone on $(0, \infty)$ too. A continuous function $f$ defined on $I$ is called an *operator convex function* on $I$ if $f(sA + (1-s)B) \leqslant sf(A) + (1-s)f(B)$ for every $0 < s < 1$ and for every pair of bounded self-adjoint operators $A$ and $B$ whose spectra are both in $I$. An *operator concave function* is likewise defined. If $I = (0, \infty)$, then $f(t)$ is operator monotone on $I$ if and only if $f(t)$ is operator concave and $f(\infty) > -\infty$ ([14], cf. [5]). This implies that every operator monotone function on $(0, \infty)$ is operator concave. Then the associated operator mean $A \,\sigma\, B$ is defined and represented as

$$A \,\sigma\, B = A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}} \tag{1.1}$$

if $A$ is invertible [7]. $\sigma$ is said to be symmetric if $A \,\sigma\, B = B \,\sigma\, A$ for every $A$, $B$. $\sigma$ is symmetric if and only if $f(t) = tf(1/t)$. When $f(t) = t^a$ $(0 < a < 1)$, the associated mean is denoted by $A \sharp_a B$ and called weighted geometric mean. In particular,

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the case of $a = \frac{1}{2}$ is the usual geometric mean and simply denoted by $A \sharp B$. The arithmetic mean $\nabla$ and the harmonic mean ! are naturally defined. It is well-known that $A \,!\, B \leqslant A \sharp B \leqslant A \nabla B$ for every $A, B \geqslant 0$; of course these are symmetric. It is well-known that $0 < A \leqslant B$ implies that $B^{-1} \sharp A \leqslant A^{-1} \sharp A = I$, but the converse does not hold.

In the recent years, geometric means of $n$-matrices are studied by many authors. Let $\mathbb{P}_m$ be the set of all $m$-by-$m$ positive definite matrices. Define $\omega = (w_1, \dots, w_n)$ be a probability vector, i.e., $w_i > 0$ for $i = 1, \dots, n$ and $\sum_{i=1}^n w_i = 1$. Let $\Delta_n$ be the set of all probability vectors. For $\omega = (w_1, \dots, w_n) \in \Delta_n$, the *Karcher mean* $\Lambda(\omega; A_1, \dots, A_n)$ of $A_1, \dots, A_n \in \mathbb{P}_m$ is characterized as the unique positive definite solution of the matrix equation [12]

$$\sum_{i=1}^{n} w_i \log\bigl(X^{-\frac{1}{2}} A_i X^{-\frac{1}{2}}\bigr) = 0.$$

If $\omega = (\frac{1}{n}, \dots, \frac{1}{n}) \in \Delta_n$, then the Karcher mean is simply written by $\Lambda(A_1, \dots, A_n)$. In the two matrices case, $A, B \in \mathbb{P}_m$, the Karcher mean coincides with the weighted geometric mean. We note that the above matrix equation is called the Karcher equation [6]. The Karcher mean inherits many properties of geometric means (see [2,12,9,3]). For instance, $\sum_{i=1}^n w_i A_i \leqslant I$ implies $\Lambda(\omega; A_1, \dots, A_n) \leqslant I$ for $\omega = (w_1, \dots, w_n) \in \Delta_n$ in [11,16].

Related to the Karcher mean, the power mean is also discussed in [10]. The power mean of $n$-matrices is inspired from the power mean of positive numbers. For $t \in [-1, 1] \setminus \{0\}$ and $\omega = (w_1, \dots, w_n) \in \Delta_n$, the power mean $P_t(\omega; A_1, \dots, A_n)$ of $A_1, \dots, A_n \in \mathbb{P}_m$ is defined as the unique positive definite solution of the matrix equation

$$\sum_{i=1}^{n} w_i (X \sharp_t A_i) = X. \tag{1.2}$$

If $\omega = (\frac{1}{n}, \dots, \frac{1}{n}) \in \Delta_n$, then the power mean is simply written by $P_t(A_1, \dots, A_n)$. It is shown in [10] that the power mean of two matrices, $A, B \in \mathbb{P}_m$, coincides with

$$P_t(1 - w, w; A, B) = A^{\frac{1}{2}} \bigl((1 - w)I + w\bigl(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\bigr)^t\bigr)^{\frac{1}{t}} A^{\frac{1}{2}}.$$

The power mean interpolates among the arithmetic, Karcher (geometric) and harmonic means. More precisely, the Karcher mean can be considered as the limit point of the power mean as $t \to 0$, it is the same situation to the number case. We will introduce the details of relations among these means in Section 3.

The aim of this paper is to investigate the converse of Loewner–Heinz inequality in the view point of operator mean. It is organized as follows: In Section 2, we shall show that if $f(\lambda B + I)^{-1} \sharp f(\lambda A + I) \leqslant I$ for all sufficiently small $\lambda \geqslant 0$, then $f(\lambda A + I) \leqslant f(\lambda B + I)$ and $A \leqslant B$. Moreover, we will deal with a symmetric operator mean. In Section 3, we will extend the results obtained in Section 2 in the case of the power means and the Karcher mean.

## 2. Operator inequality and operator mean

We begin by recalling a few results which we will need later. If $A \sharp B \leqslant I$, then $A^p \sharp B^p \leqslant I$ for all $p \geqslant 1$ [1]. Actually, $A^p \sharp B^p$ is decreasing for $p \geqslant 1$ if $A \sharp B \leqslant I$ (see Corollary 3.3 of [13]). The following well-known result for positive invertible operators is essential (see [4]):

$$\log A \leqslant \log B \quad \Longleftrightarrow \quad B^{-p} \sharp A^p \leqslant I \quad \text{for all } p \geqslant 0. \tag{2.1}$$

In this paper we deal with a non-constant operator monotone function $f(t)$ defined on a neighborhood of $t = t_0$. However we assume $t_0 = 1$ for simplicity. In this case, for every bounded self-adjoint operator $A$ the function $f(\lambda A + I)$ is well-defined for sufficiently small $\lambda$. We also note that $f'(1) > 0$.

**Theorem 1.** *Let $f(t) > 0$ be a non-constant operator monotone function defined on a neighborhood of $t = 1$ with $f(1) = 1$, and let $A$ and $B$ be bounded self-adjoint operators. Then the following are equivalent*:

(i) $A \leqslant B$,
(ii) $\lambda A + I \leqslant \lambda B + I$ *for every* $\lambda \geqslant 0$,
(iii) $f(\lambda A + I) \leqslant f(\lambda B + I)$ *for all sufficiently small* $\lambda \geqslant 0$,
(iv) $f(\lambda B + I)^{-1} \sharp f(\lambda A + I) \leqslant I$ *for all sufficiently small* $\lambda \geqslant 0$.

**Proof.** The equivalence of (i), (ii) and (iii) were shown in [15]. Since $f(\lambda B + I)$ is positive and invertible for sufficiently small $\lambda \geqslant 0$, by (2.1), (iii) gives (iv). Assume (iv). For an arbitrary $p > 0$ take $\lambda$ so that $0 < \lambda \leqslant p$. Then as mentioned above we have

$$f(\lambda B + I)^{-\frac{p}{\lambda}} \sharp f(\lambda A + I)^{\frac{p}{\lambda}} \leqslant I,$$

because $p/\lambda \geqslant 1$. We now show the next claim: