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Numerical ranges of weighted composition operators

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ABSTRACT

The operator that takes the function f to $\psi f \circ \varphi$ is a weighted composition operator. We study numerical ranges of some classes of weighted composition operators on H^2, the Hardy–Hilbert space of the unit disc. We consider the case where φ is a rotation of the unit disc and identify a class of convexoid operators. In the case of isometric weighted composition operators we give a complete classification of their numerical ranges. We also consider the inclusion of zero in the interior of the numerical range.

1. Introduction

Let φ be a holomorphic self-map of the open unit disc $\mathbb{D}$ and ψ be a holomorphic map of $\mathbb{D}$. If f is holomorphic on $\mathbb{D}$, then the operator that takes f to $\psi \cdot f \circ \varphi$ is a weighted composition operator and is denoted by $C_{\psi,\varphi}$. If z is in $\mathbb{D}$, then

$$C_{\psi,\varphi}(f)(z) = \psi(z) f\big(\varphi(z)\big).$$

In case $\psi \equiv 1$, the operator is simply called a composition operator, and is denoted by C_φ. In this work we investigate numerical ranges of weighted composition operators acting on the Hardy space H^2.

The numerical range of a bounded linear operator T on a Hilbert space $\mathcal{H}$ is the subset $W(T)$ of the complex plane given by

$$W(T) = \big\{\big\langle T(g), g\big\rangle \colon\ g \in \mathcal{H},\ \|g\| = 1\big\}. \tag{1.1}$$

Numerical ranges of (unweighted) composition operators acting on H^2 are discussed in [2,3,16].

In Section 3 we consider $C_{\psi,\varphi}$ with rotational composition maps, i.e. $\varphi(z) = e^{i\theta} z$. We identify a class of convexoid operators with rotational composition maps in Theorem 3.4. If V is an open convex set with n-fold symmetry about the origin, where $n > 1$, we prove in Theorem 3.5 that there is a weighted composition operator $C_{\psi,\varphi}$ where $\varphi(z) = e^{2\pi i/n} z$ such that $W(C_{\psi,\varphi}) = V$. In Theorem 3.13 we show that $W(C_{\psi,\varphi})$ contains such a convex, n-fold symmetric set whenever $\varphi(z) = e^{2\pi i/n} z$ and ψ is bounded.

Isometric weighted composition operators are studied in Section 4. Isometries that are not unitary operators are studied using the Wold decomposition. We also compute the numerical ranges of unitary weighted composition operators.

Inspired by Bourdon and Shapiro's work on numerical ranges of composition operators [3], in Section 5 we consider the question of when zero is in the interior of $W(C_{\psi,\varphi})$. We provide the answer for different weighted composition operators and in some cases obtain the radius of a disc centered at the origin that lies in the numerical range.

Weighted composition operators naturally appear in studies of linear operators. For example, isometries on H^p for $p \neq 2$, are weighted composition operators [7]. A composition operator on the Hardy space of the upper half-plane is similar

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to a weighted composition operator on the Hardy space of the unit disc. Hermitian weighted composition operators are investigated in [4,6] and normal weighted composition operators appear in [1]. Compact weighted composition operators are discussed in [9,10] and invertibility in [11]. These operators also play an important role in adjoints of composition operators.

2. Background material

2.1. The Hardy–Hilbert space

The set of functions analytic on $\mathbb{D}$ for which

$$\sup_{0<r<1} \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^2 \frac{d\theta}{2\pi} < \infty$$

is the Hardy–Hilbert space on the unit disc H^2. We refer to this space simply as the Hardy space. If f is in H^2, then f can be extended to the unit circle almost everywhere by taking radial limits [5, p. 10]. H^2 is a Hilbert space with the inner product

$$\langle f, g\rangle = \int_0^{2\pi} f\left(e^{i\theta}\right)\overline{g\left(e^{i\theta}\right)} \frac{d\theta}{2\pi}.$$

If f is in H^2 and $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2$. The inner product on H^2 can also be expressed as

$$\langle f, g\rangle = \sum_{n=0}^{\infty} a_n \overline{c_n},$$

where $g(z) = \sum_{n=0}^{\infty} c_n z^n$.

The *reproducing kernel* for $w \in \mathbb{D}$ is the function $K_w(z) = 1/(1 - \overline{w}z)$. Clearly $K_w \in H^2$, and if $f \in H^2$, then

$$\langle f, K_w\rangle = f(w).$$

In particular, $\|K_w\|^2 = \langle K_w, K_w\rangle = 1/(1 - |w|^2)$. Furthermore, if $C_{\psi,\varphi}$ is bounded, then

$$C^*_{\psi,\varphi}(K_w) = \overline{\psi(w)}K_{\varphi(w)}; \tag{2.1}$$

see, for example, [18, Lemma 3.2].

2.2. Notation

We use the following notation in this paper.

The closure of a subset A of the complex plane will be denoted by $Cl(A)$, the convex hull of A by $Hull(A)$, and $\overline{A}$ will be used to denote the set of complex conjugates of numbers in A.

The unit circle with center at the origin will be denoted by $\mathbb{T}$. The disc centered at a with radius r will be denoted by $D(a, r)$.

The pseudohyperbolic distance $|(a - b)/(1 - \bar{a}b)|$ between points $a, b \in \mathbb{D}$ will be denoted by $\rho(a, b)$, and $\Delta(a, r)$ is our notation for the pseudohyperbolic disc centered at a with radius r.

The space of bounded analytic functions on $\mathbb{D}$ will be denoted by H^∞.

For an operator S on H^2 we use $\sigma(S)$ to denote the spectrum of S, and $\sigma_p(S)$ for the point spectrum of S.

If φ maps the disc into itself we use φ_n to denote the nth iterate of φ i.e., φ_n is φ composed with itself n times. Also $\varphi_0(z) = z$.

2.3. Weighted composition operators

If $\psi \in H^\infty$, then it is elementary that the multiplication operator M_ψ defined by $M_\psi(f)(z) = \psi(z)f(z)$ is a bounded operator on H^2 with $\|M_\psi(f)\| \leqslant \|\psi\|_\infty\|f\|$. The composition operator C_φ, where φ is an analytic self-map of the open unit disk and $C_\varphi(f)(z) = f(\varphi(z))$, is also bounded [5, Chapter 3] on H^2. Thus, when $\psi \in H^\infty$ the operator $C_{\psi,\varphi}$ can be factored as a product of two bounded operators:

$$C_{\psi,\varphi} = M_\psi C_\varphi.$$

However, as shown by examples in [9,18], it is possible for $C_{\psi,\varphi}$ to be bounded and even compact with an unbounded ψ.