J. Math. Anal. Appl. 413 (2014) 476–481

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Finite determinacy and polynomial normal forms for diffeomorphisms near a strongly 1-resonant fixed point

Zhihua Ren [a,1], Shuo Yan [a], Jiazhong Yang [b,*,2]

[a] College of Applied Sciences, Beijing University of Technology, Beijing 100124, China
[b] School of Mathematical Sciences, Peking University, Beijing 100871, China

ARTICLE INFO

Article history:
Received 1 August 2012
Available online 21 November 2013
Submitted by P. Yao

Keywords:
Diffeomorphism
Normal form
Strongly 1-resonant

ABSTRACT

In this paper we study polynomial normal forms and smooth (C^∞) classification of one kind of diffeomorphisms which have infinite many resonant relations but are finitely determined. We derive a complete list of normal forms of all such germs with arbitrary degeneracy of their nonlinear parts.

1. Introduction and main results

Let $Diff(\mathbb{R}^n)$ be the space of all germs of C^∞ diffeomorphisms from $\mathbb{R}^n$ into $\mathbb{R}^n$ with a fixed point at the origin O. Two germs F and G in $Diff(\mathbb{R}^n)$ are called conjugating equivalent if there is a smooth change of coordinates $H : (\mathbb{R}^n, O) \longrightarrow (\mathbb{R}^n, O)$ such that $F \circ H = H \circ G$. In this paper, we shall consider the following questions. Suppose that two germs F and G in a subset $\Omega \subset Diff(\mathbb{R}^n)$, where Ω will be specified shortly, have identical Taylor series up to, say, k-jet. Then are they conjugating equivalent? If so, then what is a representation of a germ in each conjugating equivalent class? And what is the minimal possible number k, which is conventionally called the index of finite determinacy, such that the diffeomorphism is k-jet determined?

The above questions are related to the so-called finite determinacy of diffeomorphisms. The definition of finite determinacy is not new. For example, the distinguished classical Hartman–Grobman theorem says that any hyperbolic diffeomorphism is topologically 1-determined. Namely, the 1-jet of the diffeomorphism, i.e. its linear part, already sufficiently determines the topological properties of the diffeomorphism.

The finite determinacy of diffeomorphisms is characterized in [6,7]. Following Mather, we say that a formal diffeomorphism is formally k-determined if all formal diffeomorphisms with the same k-jet are formally conjugating equivalent to it. A formal diffeomorphism is called formally finitely determined if there exists an integer k such that the diffeomorphism is k-determined. Similarly, one can introduce k-determinacy in smooth category, analytic category, etc.

To describe finite determinacy, consider a germ of diffeomorphism $F \in Diff(\mathbb{R}^n)$ in a fixed coordinate system having the following form

$$F(x) = Lx + \cdots, \tag{1}$$

* Corresponding author.
E-mail addresses: zhren@bjut.edu.cn (Z. Ren), yansh08@sina.com (S. Yan), jyang@math.pku.edu.cn (J. Yang).

[1] Z. Ren is supported by grants NSFC 10601004 and NSF of Beijing 1072002.
[2] J. Yang is supported by NSFC 11271026.

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.11.036

where $L = DF(O)$ is the linear operator derived from $F(x)$, and the dots denote the nonlinear terms. We assume that the eigenvalues of L are $\lambda = (\lambda_1, \dots, \lambda_n)$.

Definition 1. (See [1,3].) Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{C}^n$. If there exists a vector $\mathbf{m} = (m_1, m_2, \dots, m_n)$, with integer component $m_i \geqslant 0$, $|\mathbf{m}| = m_1 + \cdots + m_n \geqslant 2$, such that

$$\lambda_j = \lambda^{\mathbf{m}} = \lambda_1^{m_1} \lambda_2^{m_2} \cdots \lambda_n^{m_n}, \quad \text{for some } j \in \{1, \dots, n\}, \tag{2}$$

then λ_j is said to admit a resonant relation of order $|\mathbf{m}|$.

If λ_j has the above resonant relation $\lambda_j = \lambda^{\mathbf{m}}$, then the corresponding term $x^{\mathbf{m}} e_j$ is a resonant monomial, where e_j is a coordinate carrier indicating that the occurrence of the resonant relation is in the j-th coordinate.

According to the Poincaré theorem (see Theorem 1 shortly below), if there are no resonant relations between λ, then $F(x)$ is formally linearizable. In terms of finite determinacy, this means that it is 1-determined. On the other hand, if there are resonant relations between λ, but the number of resonant relations is finite, then according to the Poincaré–Dulac theorem (see, for example, [1]), the normal form of $F(x)$ consists of at most these resonant monomials, therefore $F(x)$ is also formally k-determined, where k is at most the maximal order of the resonant relations.

Notice that either in non-resonance case or in the case of finite many resonant terms, the diffeomorphism $F(x)$ must be necessarily hyperbolic. Thus, according to the Chen theorem [5], such finite determinacy holds not only in the formal but also in the smooth category. Namely, we have the following two theorems:

Theorem 1 *(Poincaré–Dulac). (See [1,3].) A formal map with resonant linear part is formally equivalent to a map whose linear part is in Jordan normal form and whose nonlinear part consists only of resonant terms.*

Theorem 2 *(Chen). (See [1,3].) If two smooth diffeomorphisms at a hyperbolic fixed point are formally equivalent, then they are smoothly equivalent.*

Beyond the set of diffeomorphisms having finite number of resonant terms, it is natural to ask such a question: can a diffeomorphism with infinitely many resonant terms be finitely determined? To answer this question, we shall first recall the following definition.

Definition 2. (See Arnold and Il'yashenko [1].) An n-tuple $\lambda = (\lambda_1, \dots, \lambda_n) \in C^n$ is said to be 1-resonant, if there is an algebraic relation

$$\lambda^{\mathbf{p}} = \lambda_1^{p_1} \cdots \lambda_n^{p_n} = 1, \tag{3}$$

where $\mathbf{p} = (p_1, \dots, p_n) \in Z_+^n$ with $p_1 + \cdots + p_n > 0$, and for any other relation of the form $\lambda^{\mathbf{q}} = 1$ it must hold $\mathbf{q} = k\mathbf{p}$ for some $k > 0$. In particular, an n-tuple $\lambda = (\lambda_1, \dots, \lambda_n) \in C^n$ is said to be strongly 1-resonant, if it is 1-resonant and any resonant relation (2) is a trivial derivation $\lambda_i = \lambda_i \lambda^{k\mathbf{p}}$ with some integer $k > 0$.

Notice that with this definition, the germ of a diffeomorphism with $\lambda = (3, 4, 5)$ admits no resonance, $(2, 4, 5)$ has one resonant relation $\lambda_2 = \lambda_1^2$, the germ with eigenvalues $(\frac{1}{2}, 2, 3)$ is strongly 1-resonant $(\lambda_1 \cdot \lambda_2 = 1)$, while $(1, 2, 4)$ is 1-resonant but not strongly 1-resonant since the resonant relation $\lambda_3 = \lambda_2^2$ is an extraneous resonant relation, not necessarily derivable from the relation $\lambda_1 = 1$.

Now we are ready to characterize all the finitely determined diffeomorphisms:

Theorem 3. *(See Belitskii [2], Ichikawa [6].) A germ of diffeomorphism $F(x) \in Diff(\mathbb{R}^n)$ is finitely determined if and only if it has finite number of resonant terms or it is 1-resonant, and in the latter case, it is assumed not to be infinitely degenerated. Namely it does not belong to an exclusive set E of codimension infinite of the set of all germs of diffeomorphisms. Moreover, two such diffeomorphisms are smoothly equivalent provided that they are formally equivalent.*

Now we can characterize the set $\Omega \subset Diff(\mathbb{R}^n)$ mentioned at the beginning of the paper. Denote by Ω the set of all strongly 1-resonant diffeomorphisms in $Diff(\mathbb{R}^n)$. In this paper, we shall discuss the finite determinacy of germs of diffeomorphisms in Ω, giving their simplest normal form and the index of finite determinacy.

Let $F(x) \in \Omega$. Assume that its eigenvalues admit the relation (3). Then it has the following preliminary resonant normal form (notice that in strongly 1-resonant case, all eigenvalues must be necessarily different, hence there is no Jordan block in the linear part):

$$\begin{cases} \bar{x}_1 = \lambda_1 x_1 \left(1 + a_{11} x^{\mathbf{p}} + a_{12} x^{2\mathbf{p}} + \cdots\right) \\ \bar{x}_2 = \lambda_2 x_2 \left(1 + a_{21} x^{\mathbf{p}} + a_{22} x^{2\mathbf{p}} + \cdots\right) \\ \cdots \\ \bar{x}_n = \lambda_n x_n \left(1 + a_{n1} x^{\mathbf{p}} + a_{n2} x^{2\mathbf{p}} + \cdots\right), \end{cases} \tag{4}$$

where all the symbols are used in the standard way, namely, say $x^{\mathbf{p}} = x_1^{p_1} \cdots x_n^{p_n}$.