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Spectral analysis of eigenparameter dependent boundary value transmission problems

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ABSTRACT

In this paper, the singular second order differential operators are considered defined on the multi-interval. Some boundary and transmission conditions are imposed on the maximal domain functions with the spectral parameter. After constructing the differential operators associated with the boundary value transmission problems on the suitable Hilbert spaces, it is proved that these operators are the maximal dissipative operators. Finally constructing the model operators which are established with the help of the scattering functions, it is proved that all root vectors of the maximal dissipative operators are complete in the Hilbert spaces.

1. Introduction

In 1910, Weyl showed that [27] at least one of the linearly independent solutions of the singular second order differential equation is in squarly integrable space. If all solutions of the singular second order (not necessarily second order) differential equation are in squarly integrable space, then the differential expression (operator) is said to be in limit-circle case [10,20,25]. Hence, suitable boundary conditions can be given for the equation. In particular, the spectral parameter given in the differential equation may be imposed on the boundary conditions. In this situation, the corresponding operator is unusually defined, but with the operator theoretic formulation. This formulation belongs to Friedman [11]. After this formulation, a lot of selfadjoint problems have been investigated (for example see [12,13,15,23]).

On the other side, an important class of nonselfadjoint operators is the class of dissipative operators. It is well known that all eigenvalues of the dissipative operators lie in the closed upper half-plane. There are some methods to complete the spectral analysis of dissipative operators. One of them is the functional model. This method is related with the equivalence of the Lax–Phillips scattering function [17] and Sz.-Nagy–Foiaş characteristic function [19]. Lax and Phillips established the abstract scattering theory to analyze the scattering problems of acoustic waves off compact obstacles. This theory has been constructed for wave equations (hyperbolic partial differential equations). On the other hand, Sz.-Nagy and Foiaş developed the characteristic function theory of contraction operators. Adamyan and Arov [1] showed that the Lax–Phillips scattering function and the Sz.-Nagy–Foiaş characteristic function can be handled as equivalent. Pavlov used this equivalence to investigate the spectral properties of a singular second order differential equation in limit-point case [21,22]. Allahverdiev investigated such operators both in limit-point and limit-circle cases [3,8,4–6]. It should be noted that some of them contain spectral parameter in the boundary conditions.

In recent papers, the coefficients of the second order differential expression have been considered as having discontinuities in the interior of the interval. Hence, some discontinuity conditions can be given in the *transmission* points. These

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conditions are called the transmission conditions. Regular boundary value transmission problems have been investigated in many papers [2,9,16,18,26]. In these *direct* problems there are at most two transmission points [16]. In 2012, M. Shahriari et al. have investigated the *inverse* problem with a finite number of transmission points [24]. On the other hand Allahverdiev et al. have studied a singular nonselfadjoint (dissipative) boundary value transmission problem containing a spectral parameter in the boundary condition by functional model [7]. Hence, there is a gap in the regular and singular (selfadjoint and nonselfadjoint) multi-interval problems.

In this paper, singular dissipative second order differential operators with finite transmission conditions are investigated by functional model. It is proved that all eigenfunctions and associated functions (root functions) of the dissipative operators are complete in the Hilbert space $L_w^2(\Lambda)$, where $\Lambda = \bigcup_{k=1}^{n+1} \Lambda_k$ and $-\infty \leqslant \zeta_0 < \zeta_1 < \cdots < \zeta_{n+1} \leqslant \infty$. It should be noted that the results in this paper are the generalization of the results of [7].

2. Dissipative problem

The differential expression η is considered on the multi-interval $\Lambda = \bigcup_{k=1}^{n+1} \Lambda_k$, $\Lambda_k = (\zeta_{k-1}, \zeta_k)$, where $-\infty < \zeta_0 < \zeta_1 < \cdots < \zeta_{n+1} \leqslant \infty$, as

$$\eta(y) = \frac{1}{w(x)}\left[-\left(p(x)y'\right)' + q(x)y\right].$$

In this section, the following conditions are assumed to be satisfied:

i) ζ_{n+1} is the singular point for the differential expression η,
ii) the functions p, q and w are real-valued, Lebesgue measurable functions on Λ,
iii) p^{-1}, q and w are locally integrable functions on all Λ_k ($k = \overline{1, n+1} := 1, 2, \ldots, n+1$), and
iv) $w(x) > 0$ for almost all x on Λ.

In this paper, two main problems will be investigated. In this section and also in Sections 3–4, the points ζ_m ($m = \overline{0, n}$) are assumed to be regular for η and the analysis will be done for this case. In Section 5, all points ζ_k ($k = \overline{0, n+1}$) will be assumed as singular for η. Hence, now we assume that the points ζ_m are regular except the point ζ_{n+1}.

Denote by $L_w^2(\Lambda)$ the Hilbert space consisting of all complex valued functions y such that $\int_\Lambda w(x)|y(x)|^2\,dx < \infty$ with the usual inner product

$$(y, \chi) = \int_\Lambda w(x)y(x)\overline{\chi}(x)\,dx.$$

Let us consider the set $D(\Lambda)$ in $L_w^2(\Lambda)$ consisting of all functions y such that y and py' are locally absolutely continuous functions on all Λ_k ($k = \overline{1, n+1}$) and $\eta(y) \in L_w^2(\Lambda)$. For arbitrary $y, \chi \in D(\Lambda)$, one can get the following Green's formula

$$\int_\Lambda w(x)\eta(y)\overline{\chi}(x)\,dx - \int_\Lambda w(x)y(x)\overline{\eta(\chi)}\,dx = \sum_{k=1}^{n+1}[y, \chi]_{\zeta_{k-1}+}^{\zeta_k -},$$

where $[y, \chi]$ denotes the Lagrange bracket with the equalities $[y, \chi]_{\zeta_{k-1}+}^{\zeta_k -} = [y, \chi]_{\zeta_k -} - [y, \chi]_{\zeta_{k-1}+}$ and $[y, \chi]_x = y(x)\overline{(p\chi')}(x) - (py')(x)\overline{\chi}(x)$. Green's formula implies that at singular point ζ_{n+1} the value $[y, \chi]_{\zeta_{n+1}-}$ exists and is finite.

One of the main tools to investigate the singular differential equations is the Weyl theory. According to the Weyl theory [27], all solutions of the second order differential equation can belong to the squarly integrable space or one of the linearly independent solution may belong to squarly integrable space. These situations are called limit-circle case and limit-point case, respectively. It is considered that limit-circle case holds for η at singular end point ζ_{n+1} [10,20,25,27].

Now we define the real solutions of $\eta(y) = 0$ ($x \in \Lambda$) as $z(x) = \{z_1(x), z_2(x), \ldots, z_{n+1}(x)\}$ and $v(x) = \{v_1(x), v_2(x), \ldots, v_{n+1}(x)\}$, where $z_k(x)$ and $v_k(x)$ are the parts of the functions $z(x)$ and $v(x)$ defined on the interval Λ_k ($k = \overline{1, n+1}$), respectively, satisfying the conditions

$$\begin{cases} z_k(\zeta_{k-1}+) = 1, & \left(p_k z_k'\right)(\zeta_{k-1}+) = 0, \\ v_k(\zeta_{k-1}+) = 0, & \left(p_k v_k'\right)(\zeta_{k-1}+) = 1, \end{cases}$$

where $k = \overline{1, n+1}$ and $p(x) = \{p_1(x), p_2(x), \ldots, p_{n+1}(x)\}$.

For $y \in D(\Lambda)$, we consider the following boundary value transmission problem

$$-\left(p(x)y'\right)' + q(x)y = \mu w(x)y, \quad x \in \Lambda, \tag{2.1}$$

$$\left(\gamma_1 y(\zeta_0) - \gamma_2\left(py'\right)(\zeta_0)\right) - \mu\left(\gamma_1' y(\zeta_0) - \gamma_2'\left(py'\right)(\zeta_0)\right) = 0, \tag{2.2}$$

$$B_m(y) := y(\zeta_m -) - \kappa_m y(\zeta_m +) = 0, \tag{2.3}$$