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A C^*-algebra of singular integral operators with shifts admitting distinct fixed points [☆]

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ABSTRACT

Representations on Hilbert spaces for a nonlocal C^*-algebra $\mathfrak{B}$ of singular integral operators with piecewise slowly oscillating coefficients extended by a group of unitary shift operators are constructed. The group of unitary shift operators U_g in the C^*-algebra $\mathfrak{B}$ is associated with a discrete amenable group G of orientation-preserving piecewise smooth homeomorphisms $g : \mathbb{T} \to \mathbb{T}$ that acts topologically freely on $\mathbb{T}$ and admits distinct fixed points for different shifts. A C^*-algebra isomorphism of the quotient C^*-algebra $\mathfrak{B}/\mathcal{K}$, where $\mathcal{K}$ is the ideal of compact operators, onto a C^*-algebra of Fredholm symbols is constructed by applying the local-trajectory method, spectral measures and a lifting theorem. As a result, a Fredholm symbol calculus for the C^*-algebra $\mathfrak{B}$ or, equivalently, a faithful representation of the quotient C^*-algebra $\mathfrak{B}/\mathcal{K}$ on a suitable Hilbert space is constructed and a Fredholm criterion for the operators $B \in \mathfrak{B}$ is established.

1. Introduction

In this paper we deal with a nonlocal C^*-algebra $\mathfrak{B}$ generated by a C^*-algebra of singular integral operators with piecewise slowly oscillating coefficients and by a group of unitary shift operators U_G associated with a discrete amenable group G of orientation-preserving piecewise smooth homeomorphisms. The aim is to construct a *Fredholm symbol map* for the C^*-algebra $\mathfrak{B}$, that is, a representation $\Psi_{\mathfrak{B}} : \mathfrak{B} \to \mathcal{B}(\mathcal{H}_{\mathfrak{B}})$ of $\mathfrak{B}$ on a convenient Hilbert space $\mathcal{H}_{\mathfrak{B}}$ such that an operator $B \in \mathfrak{B}$ is Fredholm if and only if $\Psi_{\mathfrak{B}}(B)$ is invertible on $\mathcal{H}_{\mathfrak{B}}$. Thus, the map $\Psi_{\mathfrak{B}}$ produces a faithful representation of the quotient C^*-algebra $\mathfrak{B}/\mathcal{K}$ on the Hilbert space $\mathcal{H}_{\mathfrak{B}}$, where $\mathcal{K}$ is the ideal of compact operators.

The more complicated the structure of the set of fixed points of shifts is, the harder a Fredholm symbol calculus for $\mathfrak{B}$ is constructed and the more complicated the structure of symbols becomes. In the present paper we study the C^*-algebras $\mathfrak{B}$ with groups G acting topologically freely. Then shifts $g \in G$ can admit both common and non-common fixed points. The typical example of such groups is the solvable group G of affine mappings

$$g : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto k_g x + h_g \quad (k_g > 0,\ h_g \in \mathbb{R}),$$

where all shifts $g \in G$ have the common fixed point $x = \infty$ and, if $k_g \neq 1$, the shifts $g \in G \setminus \{e\}$ have, in general, distinct fixed points $x = h_g/(1 - k_g)$.

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The existence of non-common fixed points for shifts $g \in G \setminus \{e\}$ leads to essential difficulties in constructing a Fredholm symbol for operators $B \in \mathfrak{B}$ as compared with the case of common fixed points for all $g \in G$ studied in [4,5]. Moreover, this forced us to apply a new methodology different from those used in the previous papers [4–7] and based on the local-trajectory method and its generalizations using spectral measures (see [15,18,1,2]), which are especially effective if all shifts $g \in G \setminus \{e\}$ have the same set of fixed (or periodic) points. The new tools that allowed us to construct the desired symbol map for the C^*-algebra $\mathfrak{B}$ are a C^*-algebra modification of the lifting theorem (cf. [14, Theorem 1.8], [21, Theorem 3.3] and [25, Section 6.3]) and ideas borrowed from [20,17,22] and related to Banach algebras of singular integral operators with piecewise continuous coefficients and discrete subexponential groups of shifts (in particular, the crucial idea of dealing with finite subsets of orbits instead of whole orbits).

Let $\mathcal{B}(L^2(\mathbb{T}))$ be the C^*-algebra of all bounded linear operators acting on the space $L^2(\mathbb{T})$, where $\mathbb{T}$ is the unit circle in $\mathbb{C}$ oriented anticlockwise; $PSO(\mathbb{T})$ be the C^*-algebra of piecewise slowly oscillating functions on $\mathbb{T}$ defined in Subsection 2.1; G be a discrete amenable group of orientation-preserving piecewise smooth homeomorphisms $g : \mathbb{T} \to \mathbb{T}$ possessing derivatives g' with at most finite sets of discontinuities, and let G act on $\mathbb{T}$ topologically freely, that is, for every finite set $G_0 \subset G \setminus \{e\}$, where e is the unit of G, and every open set $V \subset \mathbb{T}$ there exists a point $\tau \in V$ such that $g(\tau) \neq \tau$ for every $g \in G_0$ (cf. [1,18]). Suppose G acts on $\mathbb{T}$ from the right: $(g_1 g_2)(t) = g_2(g_1(t))$ for all $g_1, g_2 \in G$ and all $t \in \mathbb{T}$.

We study the C^*-subalgebra

$$\mathfrak{B} := \operatorname{alg}\big(PSO(\mathbb{T}), S_{\mathbb{T}}, U_G\big) \tag{1.1}$$

of $\mathcal{B}(L^2(\mathbb{T}))$ generated by all multiplication operators aI with $a \in PSO(\mathbb{T})$, by the Cauchy singular integral operator $S_{\mathbb{T}}$ defined by

$$(S_{\mathbb{T}}\varphi)(t) := \lim_{\varepsilon \to 0} \frac{1}{\pi i} \int\limits_{\mathbb{T}\setminus\mathbb{T}(t,\varepsilon)} \frac{\varphi(\tau)}{\tau - t}\, d\tau, \qquad \mathbb{T}(t,\varepsilon) = \big\{\tau \in \mathbb{T}\colon |\tau - t| < \varepsilon\big\}, \quad t \in \mathbb{T}, \tag{1.2}$$

and by the group $U_G := \{U_g\colon g \in G\}$ of unitary weighted shift operators U_g given by

$$(U_g\varphi)(t) := \big|g'(t)\big|^{1/2} \varphi\big(g(t)\big) \quad \text{for } t \in \mathbb{T}. \tag{1.3}$$

The paper is organized as follows. Section 2 contains preliminaries: definition of the commutative C^*-algebra $PSO(\mathbb{T})$ of piecewise slowly oscillating functions on $\mathbb{T}$ and a description of its maximal ideal space $M(PSO(\mathbb{T}))$, as well as a Fredholm symbol and a Fredholm criterion for the C^*-algebra

$$\mathfrak{A} := \operatorname{alg}\big(PSO(\mathbb{T}), S_{\mathbb{T}}\big) \subset \mathfrak{B} \tag{1.4}$$

generated by the operator $S_{\mathbb{T}}$ and all operators aI with $a \in PSO(\mathbb{T})$. Section 2 also contains a description of a spectral measure associated with a central subalgebra of the C^*-algebra $\mathfrak{B}/\mathcal{K}$.

Section 3 contains the main results of the paper: a representation $\Psi_{\mathfrak{B}}$ of the C^*-algebra $\mathfrak{B}$ on a Hilbert space $\mathcal{H}_{\mathfrak{B}}$ with $\operatorname{Ker}\Psi_{\mathfrak{B}} = \mathcal{K}$, where

$$\Psi_{\mathfrak{B}} = \Big(\bigoplus_{w \in W_{\mathbb{T}}} \Psi_w\Big) \oplus \Big(\bigoplus_{w \in W_{\mathbb{T}}^0} \Psi_w^0\Big), \qquad \mathcal{H}_{\mathfrak{B}} := \Big(\bigoplus_{w \in W_{\mathbb{T}}} \mathcal{H}_w\Big) \oplus \Big(\bigoplus_{w \in W_{\mathbb{T}}^0} \mathcal{H}_w^0\Big), \tag{1.5}$$

$W_{\mathbb{T}}$ is the set of all G-orbits of points $t \in \mathbb{T}$ and $W_{\mathbb{T}}^0$ is the set of G-orbits containing more than one point. As a result, a Fredholm symbol map for $\mathfrak{B}$ is obtained (Theorem 3.1) and a Fredholm criterion for the operators $B \in \mathfrak{B}$ in terms of their Fredholm symbols is established (Theorem 3.2).

In Section 4 we give an example of a Fredholm singular integral operator with shift, which illustrates the conditions of Theorem 3.2 in an explicit form.

Section 5 contains the main tools for studying the C^*-algebra $\mathfrak{B}$: a suitable version of the local-trajectory method (Theorem 5.1), a lifting theorem for C^*-algebras (Theorem 5.2) and its corollary providing sufficient Fredholm conditions (Theorem 5.3).

In Section 6, applying the local-trajectory method, we study the invertibility of functional operators that constitute the C^*-algebra

$$\mathcal{A} := \operatorname{alg}\big(PSO(\mathbb{T}), U_G\big) \subset \mathfrak{B}. \tag{1.6}$$

In Section 7, making use of results for the C^*-algebra (1.6), we prove that every operator $B \in \mathfrak{B}$ for considered groups G is of the form

$$B = A^+ P_{\mathbb{T}}^+ + A^- P_{\mathbb{T}}^- + H_B, \tag{1.7}$$

where $A^\pm \in \mathcal{A}$, $P_{\mathbb{T}}^\pm := (I \pm S_{\mathbb{T}})/2$, $H_B \in \mathfrak{H}$, and $\mathfrak{H}$ is the closed two-sided ideal in $\mathfrak{B}$ generated by all commutators $[aI, S_{\mathbb{T}}]$ and $[U_g, S_{\mathbb{T}}]$, where $a \in PSO(\mathbb{T})$ and $g \in G$ (Theorem 7.2). Using this decomposition we prove in Section 7 that