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Spectral analysis of non-commutative harmonic oscillators: The lowest eigenvalue and no crossing

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ABSTRACT

The lowest eigenvalue of non-commutative harmonic oscillators $Q(\alpha,\beta)$ ($\alpha>0$, $\beta>0$, $\alpha\beta>1$) is studied. It is shown that $Q(\alpha,\beta)$ can be decomposed into four self-adjoint operators,

$$Q(\alpha,\beta)=\bigoplus_{\sigma=\pm,\mathrm{p}=1,2} Q_{\sigma\mathrm{p}},$$

and all the eigenvalues of each operator $Q_{\sigma\mathrm{p}}$ are simple. We show that the lowest eigenvalue of $Q(\alpha,\beta)$ is simple whenever $\alpha\neq\beta$. Furthermore a Jacobi matrix representation of $Q_{\sigma\mathrm{p}}$ is given and spectrum of $Q_{\sigma\mathrm{p}}$ is considered numerically.

1. Introduction

The non-commutative harmonic oscillator is introduced by A. Parmeggiani and M. Wakayama [8–10] as a non-commutative extension of harmonic oscillators. We also refer to [7] which is a first account about non-commutative harmonic oscillators and of their spectral properties. It is defined by

$$Q=Q(\alpha,\beta)=A\otimes\left(-\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{2}x^2\right)+J\otimes\left(x\frac{d}{dx}+\frac{1}{2}\right), \tag{1.1}$$

as an operator in $\mathcal{H}=\mathbb{C}^2\otimes L^2(\mathbb{R})$. Here $A,J\in\mathrm{Mat}_2(\mathbb{R})$, A is positive definite symmetric, and J skew-symmetric. Furthermore $A+iJ$ is positive definite. It is shown in [9,10] that A and J can be assumed to be $A=\left(\begin{smallmatrix}\alpha&0\\0&\beta\end{smallmatrix}\right)$, $J=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$, and α and β satisfy

$$\alpha>0,\qquad\beta>0,\qquad\alpha\beta>1. \tag{1.2}$$

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We fix A and J as above, and throughout this paper we assume (1.2). Under (1.2), Q is self-adjoint on the domain $D(Q) = \mathbb{C}^2 \otimes (D(d^2/dx^2) \cap D(x^2))$ and has purely discrete spectrum $E_0 \leqslant E_1 \leqslant E_2 \leqslant \cdots \nearrow \infty$. When $\alpha = \beta$, $Q(\alpha, \beta)$ is equivalent to the direct sum of a harmonic oscillator. Then $E_j = E_{j+1} = \frac{1}{2}(1 + j)\sqrt{\alpha^2 - 1}$ for $j = 0, 2, 4, \ldots$. In the case of $\alpha \neq \beta$, however, the spectrum of $Q(\alpha, \beta)$ is nontrivial, and exploring properties of the spectrum is the main purpose of the present paper.

An eigenvector associated with the lowest eigenvalue $E = E_0$ is called a ground state in this paper. A long-standing problem concerning eigenvalues of $Q(\alpha, \beta)$ is to determine their multiplicity explicitly. Let $\alpha \neq \beta$. Let $E_n = E_n(\alpha, \beta)$ denote the n-th eigenvalue of $Q(\alpha, \beta)$. The map $c_n : (\alpha, \beta) \mapsto E_n(\alpha, \beta) \in \mathbb{R}$ is called an eigenvalue-curve. To consider the multiplicity of eigenvalues is reduced to considering crossing or no crossing of eigenvalue-curves.

We state a short history concerning studies of the multiplicity of eigenvalues of Q. In [10] it is shown that the multiplicity of any eigenvalues of Q is at most three and an alternative proof is given in [5]. At a numerical level it is found in [4] that eigenvalue-curves cross at some points but the lowest eigenvalue is simple. The multiplicity of eigenvalues of Q is also considered in [3], where it is derived that

$$\left(n - \frac{1}{2}\right) \min\{\alpha, \beta\} \sqrt{\frac{\alpha\beta - 1}{\alpha\beta}} \leqslant E_{2n-1} \leqslant E_{2n} \leqslant \left(n - \frac{1}{2}\right) \max\{\alpha, \beta\} \sqrt{\frac{\alpha\beta - 1}{\alpha\beta}}$$

for $n = 1, 2, 3, \ldots$. From this we can see that the multiplicity of E is at most two if $\beta < 3\alpha$ or $\alpha < 3\beta$. In [6] it is shown that E is simple but for sufficiently large $\alpha\beta$. Furthermore in [2] it is proven that the lowest eigenvalue is at most two and all the ground states are even for $(\alpha, \beta) \in D_{\sqrt{2}}$, where $D_{\sqrt{2}} = \{(\alpha, \beta) \mid \alpha, \beta > \sqrt{2}\}$, and it is also shown that E is simple for $(\alpha, \beta) \in D$ for some subset $D \subset D_{\sqrt{2}}$. Recently Wakayama [11] breaks through in studying the multiplicity of E, in that he proves that if all the ground states are even, then E is simple whenever $\alpha \neq \beta$. Combining [11] with [2], it is immediate to see that E is simple for $(\alpha, \beta) \in D_{\sqrt{2}}$.

In this paper we settle down the question concerning the multiplicity of the lowest eigenvalue of Q, i.e., we prove that E is simple for all values of α and β ($\alpha \neq \beta$), see Theorem 3.1. Moreover no crossing between eigenvalue-curves associated with an odd eigenvector and an even eigenvector can occur, as proved in Corollary 5.2.

This paper is organized as follows. In Section 2, we decompose $Q(\alpha, \beta)$ into four self-adjoint operators: $Q(\alpha, \beta) = \bigoplus_{\sigma=\pm, \mathrm{p}=1,2} Q_{\sigma\mathrm{p}}$. It is shown that each $Q_{\sigma\mathrm{p}}$ is equivalent to some Jacobi matrix $\widehat{Q}_{\sigma\mathrm{p}}$, and all the eigenvalues of $Q_{\sigma\mathrm{p}}$ are simple. In Section 3, we show that the lowest eigenvalue of $Q(\alpha, \beta)$ is simple. In Section 4, we construct a unitary transformation $U_{\sigma\mathrm{p}}$ such that $e^{-tU_{\sigma\mathrm{p}}^{-1} Q_{\sigma\mathrm{p}} U_{\sigma\mathrm{p}}}$ is positivity improving, and it is shown that the ground state is in a positive cone. In Section 5, we show that $\widehat{Q}_{-\mathrm{p}} - \widehat{Q}_{+\mathrm{p}} \geqslant \Delta(\alpha, \beta)$, $\mathrm{p} = 1, 2$, for some $\Delta(\alpha, \beta)$. In particular, if $\Delta(\alpha, \beta) > 0$, then there is no crossing between the n-th eigenvalue-curve of $Q_{-\mathrm{p}}$ and that of $Q_{+\mathrm{p}}$. In Section 6, we show some numerical results.

2. Decomposition of $Q(\alpha, \beta)$ and Jacobi matrix

2.1. Decomposition of $Q(\alpha, \beta)$

Let $a = \frac{1}{\sqrt{2}}(x + \frac{d}{dx})$ and $a^* = \frac{1}{\sqrt{2}}(x - \frac{d}{dx})$ be the annihilation operator and the creation operators, respectively. In terms of a and a^*, Q can be expressed as

$$Q = A\left(a^* a + \frac{1}{2}\right) + \frac{J}{2}(aa - a^* a^*). \tag{2.1}$$

Let $\mathcal{H}_+$ (resp. $\mathcal{H}_-$) be the set of even (resp. odd) functions in $\mathcal{H}$, and P_+ (resp. P_-) be the orthogonal projection onto $\mathcal{H}_+$ (resp. $\mathcal{H}_-$). Let $|n\rangle$ be the n-th normalized eigenvector of $a^* a$, i.e., $|n\rangle = \frac{1}{\sqrt{n!}}(a^*)^n |0\rangle$