J. Math. Anal. Appl. 415 (2014) 661–676

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A new necessary and sufficient condition for the strong duality and the infinite dimensional Lagrange multiplier rule

Antonino Maugeri *, Daniele Puglisi

Department of Mathematics and Computer Sciences, University of Catania, Catania, 95125, Italy

ARTICLE INFO

Article history:
Received 8 April 2013
Available online 23 January 2014
Submitted by A. Daniilidis

Keywords:
Strong duality
Lagrange dual problem
Elastic–plastic torsion problem
Continuum model of transportation
Evolutionary illumination and visibility problem

ABSTRACT

Throughout this paper, the authors introduce a new condition, defined by *Assumption* S', which establishes a necessary and sufficient condition for the validity of the strong duality between a convex optimization problem and its Lagrange dual. This work will be focused on the context of emptiness of the interior of the ordering cone and convexity of the equality constraints. Moreover, this new condition will be necessary and sufficient for the infinite dimensional Lagrange multiplier rule. This new principle will find application to the elastic–plastic torsion problem, to the continuum model of transportation and to a problem with quadratic equality constraint with connected to evolutionary illumination and visibility problems.

1. Introduction

The aim of this paper is to show a new necessary and sufficient condition involving the derivatives of the mappings which define the optimization problem, ensuring the validity of the strong duality between a wide class of convex optimization problems and its Lagrange dual. Such work will include the case of emptiness of the interior ordering cone defined by the sign constraints.

This new condition allows us to remove the assumption of affine-linearity on the equality constraints as required for the applications of *Assumption S* introduced in the papers [7,10–12], replacing the affinity with the existence of directional derivatives and Gâteaux differentiability of a constraint map. Moreover, we will show that this new condition, defined by *Assumption* S', it is a necessary condition for the strong duality and a necessary and sufficient condition to the infinite dimensional Lagrange multiplier rule. For other approaches for the strong duality see also [4,2,3,21–23,28,27,26].

Throughout this paper, we shall use the notation $\mathbb{R}^+ = \{\lambda \in \mathbb{R}\colon\ \lambda \geqslant 0\}$, $\mathbb{R}^{++} = \{\lambda \in \mathbb{R}\colon\ \lambda > 0\}$; mutatis mutandis for $\mathbb{R}^-$ and $\mathbb{R}^{--}$.

In detail, the convex optimization problem we are concerned with is the following:

* Corresponding author.
E-mail addresses: maugeri@dmi.unict.it (A. Maugeri), dpuglisi@dmi.unict.it (D. Puglisi).

http://dx.doi.org/10.1016/j.jmaa.2014.01.032

Let $(X,\|\cdot\|_X)$, $(Z,\|\cdot\|_Z)$ be real Banach spaces, let $(Y,\|\cdot\|_Y)$ be a real Banach space with an ordering convex cone C (i.e., for $x,y\in Y$, we say that $x\leqslant_C y$ if and only if $y-x\in C$). Let S be a convex subset of X, let $f:S\to\mathbb{R}$ be a given functional and $g:S\to Y$ and $h:S\to Z$ be two given mappings. Setting

$$K=\{x\in S:\ g(x)\in -C,\ h(x)=\theta_Z\},$$

we consider the *optimization problem*

$$\text{find } x_0\in K \text{ such that } f(x_0)=\min_{x\in K} f(x) \tag{1.1}$$

and the *Lagrange dual problem*

$$\max_{\substack{u\in C^+\\ v\in Z^*}} \inf_{x\in S}\big[f(x)+\langle u,g(x)\rangle+\langle v,h(x)\rangle\big], \tag{1.2}$$

where

$$C^+=\{y^*\in Y^*:\ \langle y^*,c\rangle\geqslant 0,\ \forall c\in C\}$$

is the *dual cone of* C.

In the previous papers [7,10,11,16,19,13] the authors consider the so called *Assumption S* in order to prove the strong duality between the convex optimization problem in an infinite dimensional space and its Lagrange dual problem seen above.

The *Assumption S* is the following. Let us first recall that for a subset $C\subseteq X$ and $x\in X$ the *tangent cone* to C at x is defined as

$$T_C(x)=\Big\{y\in X:\ y=\lim_{n\to\infty}\lambda_n(x_n-x),\ \lambda_n>0,\ x_n\in C,\ \lim_{n\to\infty}x_n=x\Big\}.$$

If $x\in cl\,C$ (the closure of C) and C is convex, we have

$$T_C(x)=cl\ cone\big(C-\{x\}\big),$$

where the $cone\,A=\{\lambda x:\ x\in A,\ \lambda\in\mathbb{R}^+\}$ denotes the cone hull of a general subset A of the space.

Definition 1.1. Given the mappings f, g, h and the set K as above, we say that *Assumption S* is fulfilled at a point $x_0\in K$ if and only if

$$T_{\widetilde{M}}(0,\theta_Y,\theta_Z)\cap\big(\mathbb{R}^{--}\times\theta_Y\times\theta_Z\big)=\emptyset$$

where

$$\widetilde{M}=\big\{\big(f(x)-f(x_0)+\alpha,g(x)+y,h(x)\big):\ x\in S\setminus K,\ \alpha\geqslant 0,\ y\in C\big\}.$$

Let us recall the main theorem on strong duality based on *Assumption S*.

Theorem 1.2. *(See [11].) Assume that the functions $f:S\to\mathbb{R}$, $g:S\to Y$ are convex and that $h:S\to Z$ is an affine-linear mapping. Assume that the* Assumption S *is fulfilled at the optimal solution $x_0\in K$ of the problem (1.1). Then also problem (1.2) is solvable and if $\bar u\in C^+$, $\bar v\in Z^*$ are optimal solutions to (1.2), we have*