J. Math. Anal. Appl. 415 (2014) 803–815

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On removable sets for convex functions ☆

Dušan Pokorný *, Martin Rmoutil

Charles University, Faculty of Mathematics and Physics, Sokolovská 83, 186 75 Praha 8 Karlín, Czech Republic

ARTICLE INFO

Article history:
Received 3 September 2013
Available online 15 February 2014
Submitted by B.S. Thomson

Keywords:
Convex function
Locally convex function
Intervally thin set
c-removable set
Convex extension
Separately convex function

ABSTRACT

In the present article we provide a sufficient condition for a closed set $F \in \mathbb{R}^d$ to have the following property which we call c-removability: Whenever a continuous function $f : \mathbb{R}^d \to \mathbb{R}$ is locally convex on the complement of F, it is convex on the whole $\mathbb{R}^d$. We also prove that no generalized rectangle of positive Lebesgue measure in $\mathbb{R}^2$ is c-removable. Our results also answer the following question asked in an article by Jacek Tabor and Józef Tabor (2010) [5]: Assume the closed set $F \subset \mathbb{R}^d$ is such that any locally convex function defined on $\mathbb{R}^d \setminus F$ has a unique convex extension on $\mathbb{R}^d$. Is F necessarily intervally thin (a notion of smallness of sets defined by their "essential transparency" in every direction)? We prove the answer is negative by finding a counterexample in $\mathbb{R}^2$.

1. Introduction

The present article is mostly motivated by the work [5] about negligible sets for convexity of functions in $\mathbb{R}^d$, where an interesting open problem was raised. We shall need the following notion introduced in [5].

A set $A \subset \mathbb{R}^d$ is called *intervally thin* if for any $x, y \in \mathbb{R}^d$ and any $\varepsilon > 0$ there exist $x' \in B(x, \varepsilon)$ and $y' \in B(y, \varepsilon)$ such that $[x', y'] \cap A = \emptyset$.

Problem TT. Let $A \subset \mathbb{R}^n$ be closed. Suppose that for an arbitrary open set U containing A every locally convex function $f : U \setminus A \to \mathbb{R}$ has a unique extension on U. Is it then necessarily true that A is intervally thin?

Arguably our main result is that the answer to this question is negative. Example 4.2 and Remark 4.3 provide a closed set K which is not intervally thin, but which enjoys the "unique extension property for convex functions" (UEP) from Problem TT. We took the liberty of calling this set K "the Holey Devil's

☆ D. Pokorný was supported by a cooperation grant of the Czech and the German science foundation, GAČR Project No. P201/10/J039, M. Rmoutil by the Grant No. 710812 of the Grant Agency of the Charles University in Prague.

* Corresponding author.
E-mail addresses: dpokorny@karlin.mff.cuni.cz (D. Pokorný), caj@rmail.cz (M. Rmoutil).

http://dx.doi.org/10.1016/j.jmaa.2014.02.014
0022-247X/

Staircase" since it is the graph of the classical Cantor function (the Devil's Staircase) minus all the horizontal open line segments contained in the graph (in other words, it is the graph of the restriction of the Cantor function to the Cantor set).

One can readily verify that the Holey Devil's Staircase is not intervally thin. It is enough to consider the last intersection of the graph of the Cantor function with any line segment with endpoints in $(-\infty, 0) \times (0, \frac{1}{2})$ and $(1, \infty) \times (\frac{1}{2}, 1)$; clearly, this intersection is an element of K.

To prove that K has the UEP, is considerably more difficult and our effort in this direction has inspired a large part of this article.

The main result of [5] is essentially the following theorem. Note that since we restrict our attention to convex functions (as opposed to ω-semiconvex functions studied in [5]), we change the formulation of the theorem accordingly:

Theorem TT. *Let U be an open subset of $\mathbb{R}^d$ and let A be a closed intervally thin subset of U. Let $f : U \setminus A \to \mathbb{R}$ be a locally convex function. Then f has a unique locally convex extension on U.*

The proof of this theorem consists of two principal steps:

(1) First, one proves that there is a unique continuous extension; this is the more difficult part.
(2) Once one has the continuous extension, it is then easy to prove that it is convex.

Our aim is to apply this scheme to our set K. It turns out that in this case the easier step is (1); we only need a simple generalization of the corresponding theorem from [5]—which we have in Lemma 4.1.

Performing step (2) for K is the crucial part and it motivates the introduction of c-removable sets with the consequent natural question: Which sets are c-removable?

Definition. We say that a closed set $A \subset \mathbb{R}^d$ is *c-removable* if the following is true: Every real function f on $\mathbb{R}^d$ is convex whenever it is continuous on $\mathbb{R}^d$ and locally convex on $\mathbb{R}^d \setminus A$.

A consequence of Theorem TT is that all closed intervally thin sets are c-removable, but this fact does not help us. In $\mathbb{R}^2$ we were able to find a sufficient condition more general than interval thinness which covers also the case of our set K:

Proposition 1. *Let $K \subset \mathbb{R}^2$ be compact and intervally thin in two different directions. Assume that for a dense set of line segments $L \subset \mathbb{R}^2$ the cardinality of $K \cap L$ is at most countable. Then K is c-removable.*

Here interval thinness of K in a direction means that to any given line segment in that direction we can find arbitrarily close line segments contained in the complement of the set K. It is not difficult to see that for any closed set K intervally thin in a direction v, any continuous function which is locally convex outside K is necessarily convex on all lines parallel to v. Hence, the assumption of interval thinness of K in two directions ensures that our function is convex in those two directions (i.e. is essentially separately convex) which we can use further in the proof—the key Lemma 3.1 tells us that a separately convex function cannot "have a concave angle" on any line.

The condition from the proposition may seem rather artificial, but it emerges quite naturally from our method of the proof. What is more, it is easily seen to be more general than interval thinness and is fulfilled by K. (Hence, the Holey Devil's Staircase is c-removable.) However, we were not able to generalize this condition to higher dimensions; instead, we used the geometric measure theory to obtain the following theorem which in $\mathbb{R}^2$ is strictly weaker than Proposition 1.