J. Math. Anal. Appl. 415 (2014) 846–872

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Detection of multilayered media in the acoustic waveguide

Fadhel Al-Musallam [a,b,*], Amin Boumenir [c,1], Mourad Sini [a,2]

[a] RICAM, Austrian Academy of Sciences, Altenbergerstrasse 69, A-4040, Linz, Austria
[b] Department of Mathematics, Kuwait University, P.O. Box 13060, Safat, Kuwait
[c] Department of Mathematics, University of West Georgia, 1601 Maple Street, Carrollton, GA 30118, United States

ARTICLE INFO

Article history:
Received 1 November 2012
Available online 8 February 2014
Submitted by H. Kang

Keywords:
Inverse scattering problems
Waveguide
Inverse spectral problems

ABSTRACT

This work is concerned with the inverse problem for ocean acoustics modeled by a multilayered waveguide with a finite depth. We provide explicit formulae to locate the layers, including the seabed, and reconstruct the speed of sound and the densities in each layer from measurements collected on the surface of the waveguide. We proceed in two steps. First, we use Gaussian type excitations on the upper surface of the waveguide and then from the corresponding scattered fields, collected on the same surface, we recover the boundary spectral data of the related 1D spectral problem. Second, from these spectral data, we reconstruct the values of the normal derivatives of the singular solutions, of the original waveguide problem, on that upper surface. Finally, we derive formulae to reconstruct the layers from these values based on the asymptotic expansion of these singular solutions in terms of the source points.

1. Introduction

1.1. The model

The propagation of acoustic waves in the waveguide $\Omega := \mathbb{R}^2 \times (L, 0)$, $L < 0$, is governed by the following model, see [6]:

$$\begin{cases} \nabla \cdot a\nabla u + \kappa^2 n u = 0 & \text{in } \Omega, \\ u = f & \text{on } x_3 = 0, \\ a\frac{\partial}{\partial x_3} u = 0 & \text{on } x_3 = L \end{cases} \tag{1.1}$$

* Corresponding author.
E-mail addresses: musallam@sci.kuniv.edu.kw (F. Al-Musallam), boumenir@westga.edu (A. Boumenir), mourad.sini@oeaw.ac.at (M. Sini).

[1] Current address: Department of Mathematics, Kuwait University, P.O. Box 13060, Safat, Kuwait.
[2] Partially supported by the Austrian Science Fund (FWF): P22341-N18.

http://dx.doi.org/10.1016/j.jmaa.2014.02.002

where $a(x) := \frac{1}{\rho(x)}$ and $n(x) := c^2(x)\rho(x)$. Here ρ is the density, c is the speed of sound and κ is the frequency of the propagation which is assumed to be constant and fixed.

We are interested in stratified waveguides, that is the coefficients a and n are assumed to satisfy the properties:

- $a(x) := a(x_3) := \sum_{i=1}^{M} a_i\chi_i$, where a_i, $i = 1, \ldots, M$ are positive constants and
- $n(x) := n(x_3) := \sum_{i=1}^{M} n_i\chi_i$, where n_i, $i = 1, \ldots, M$ are positive constants

with

$$\chi_i := \begin{cases} 1 & \text{in } (\bar{z}_{i-1}, \bar{z}_i), \\ 0 & \text{in } (L, 0) \setminus (\bar{z}_{i-1}, \bar{z}_i), \end{cases} \tag{1.2}$$

$i = 1, \ldots, M$, $\bar{z}_0 = 0$ and $\bar{z}_M = L$. Sometimes we use the notation $x := (x', x_3)$ for $x \in \mathbb{R}^3$ where $x' := (x_1, x_2)$.

The forward problem (1.1) is well posed, see Section 4. Precisely, if $f \in H^2(\mathbb{R}^2)$ with compact support or with an exponential decay at infinity, then (1.1) has one and only one solution $u \in H^1_{loc}(\Omega)$ satisfying appropriate radiation conditions at infinity in the horizontal directions of the waveguide. Hence $a\frac{\partial}{\partial x_3}u(x', 0)$ is in $H^{-\frac{1}{2}}_{loc}(\mathbb{R}^2)$ and we have the following estimate

$$\left\| a\frac{\partial}{\partial x_3}u(x', 0)\right\|_{H^{-\frac{1}{2}}_{loc}(\mathbb{R}^2)} \leqslant C\|f\|_{H^2(\mathbb{R}^2)} \tag{1.3}$$

with a positive constant C independent on f.

1.2. *The inverse problem*

In this work, we are concerned with the following inverse problem: *reconstruct the vectors*

$$(a_i)_{i=1}^M, \qquad (n_i)_{i=1}^M \quad \text{and} \quad (\bar{z}_i)_{i=1}^M$$

from the measurements $a\frac{\partial}{\partial x_3}u(x', 0)$, *for* $x' \in \mathbb{R}^2$, *corresponding to finitely many excitations* f.

In other words, we aim at localizing the layers and reconstruct the density and the speed of sound in each layer from finitely many measurements on the surface.

A classical idea is to solve such inverse problems for stratified media by reducing the problem to 1D inverse spectral problems. There is an extensive literature on these problems. Instead of reviewing the known results, we recall some of the most popular ideas proposed to solve these problems. We mention the asymptotic expansion technique used for the first time by Borg and then simplified by Levinson at the end of the 40s, see [4,13]. The second approach is the integral equation method by Gel'fand and Levitan introduced in the 50s for solving Sturm–Liouville inverse spectral problems, see [11]. During the period from the 50s till the 80s these two approaches have been extensively studied by many authors, see Refs. [14,16,20] for more details on these methods and the related results till mid 80s. The third approach is the so-called the C-property by Ramm, see [22]. The fourth approach is the boundary control method (BC method in short) by Belichev and Kurylev introduced in the mid 80s, see [3] for the original version and [12] for a different presentation. It is worth mentioning that two of them (the Gel'fand–Levitan method and the boundary control method) are constructive. In addition, the first three methods are designed to solve the problem for equations of the standard or normal form, i.e. the coefficient a is taken to be smooth enough so that we can reduce the equation in (1.4) to the form $\frac{d^2}{dy^2}v + qv = \lambda v$. The boundary control method has been generalized to the general Sturm–Liouville problem (1.4) with discontinuous coefficients a in [24]. So, in principle this